Some Results of The Cyclic Decomposition of The Rational Valued Character Table of Some Finite Groups

By

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Abstract:

The destination of this labor's is to present and study the cyclic decomposition of the rational-valued character schedule matrix of some finite groups which are (Z_{st} , Z_{str} , $Z \times C_2$ and $Z_{str} \times C_2$),where $s > t > r > 2$ are distinct prime numbers .

Keywords:-

Finite groups Z_n , cyclic decomposition , character tables of finite groups.

1. Introduction:

Many researchers such as Hussein . H . A [2] , Mohamed . S .K [5] , Rajaa . H .A[7]and Shabani . H , Ashrafi .A . R and Ghorbani . M [8], they have presented and studied the topic of the rational.. character table and the cyclic decomposition of some finite groups . Paper goal is to present the cyclic decomposition of the rational.. valued character matrix table of finite groups Z_{st} and Z_{str} of order st and str respectively, where $s > t > r$ are distinct prime numbers.

Also, we introduce the rational.. valued character schedule matrix and cyclic decomposition of the groups ($Z_{st} \times C_2$) and ($Z_{str} \times C_2$) , by using row and column operations to determine the diagonal matrix of these rational-valued character table matrices .

2-Preliminaries:-

Some notions and theories that we need in this paper , we will introduce in this section **.**

Definition (2-1),[4]:

The character whose elements belong to Z such that $\mathcal{Q}(t) \in \mathrm{Z}, \forall t \in \mathcal{K}$ is called **the rational valued character** $\mathcal{Q}$ of $\mathcal{K}$.

Definition(2-3),[5]:

Let $\mathcal{K}$ be finite group . Then $u \times u$ matrix, where the columns represent Γ-classes and its rows represent elements of each rational values characters of $\mathcal{K}$ is called **the rational.. valued characters table of $\mathcal{K}$** and is symbolizes it by $\equiv^{*}$ ($\mathcal{K}$) or QCT($\mathcal{K}$) .

Definition (2-4), [3]:-

If a matrix $\mathcal{N}$ whose elements in $\mathcal{R}$, where $\mathcal{R}$ principal integral domain is identical to matrix $\mathcal{D}$= Diag{ s_1 , s_2 , ..., s_t ,0 ,0 , ..., 0} such that s_j / s_{j+1} for $1 \leq j \leq t$, then $\mathcal{D}$ is called **the invariant factor matrix** of $\mathcal{N}$ and $\mathbf{s_1}$ **,** $\mathbf{s_2}$ **, ...,** $\mathbf{s}_t$ the invariant factor of $\mathcal{N}$.

Theorem (2-5),[6]:

If $\mathcal{R}$ is principal domain and T, Q are non-singular matrices of degree ℓ and j respectively over $\mathcal{R}$. Then $\mathcal{D}$(T⊗Q)=$\mathcal{D}$(T)⊗$\mathcal{D}$(Q).Where $\mathcal{D}$(T) and $\mathcal{D}$(Q) are matrices of invariant factor of T and Q respectively.

Remark(2-6),[6]:

$$\mathrm{OCT}(C_2) = (\equiv^*(\mathrm{C}_2)) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

Theorem(2-7),[6]:

If q is a prime number, then $\mathcal{D}(\equiv^*(\mathrm{C}_{\mathrm{q}^{\mathrm{k}}}))$=Diag{$\mathrm{q}^{\mathrm{k}}$, $\mathrm{q}^{\mathrm{k}-1}$, ⋯⋯, q,1}.

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Theorem(2-8),[2]:

If $\mathcal{K}_1$, $\mathcal{K}_2$ are two groups of the orders ℓ_1 ,$\ell_2$respectively where g.c. d(ℓ_1 ,ℓ_2)=1 ,for any $\chi \in \mathrm{Irr}(\mathcal{K}_1)$ and $\psi \in \mathrm{Irr}(\mathcal{K}_2)$ such that g. c. d(m(χ), ψ(1) $\mathbb{Q}(\psi):\mathbb{Q}\,|$)=g. c. d(mψ), χ(1) $\mathbb{Q}(\chi):\mathbb{Q}\,|$)=1,then $\equiv^*(\mathcal{K}_1 \times \mathcal{K}_2) = (\equiv^*(\mathcal{K}_1)) \otimes (\equiv^*(\mathcal{K}_2))$.

Theorem (2-9),[2]:

(i) If s and t are two prime numbers such that $s > t$, then ($\equiv^*(Z_{st})$)=

QCT(Z_{st})	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$
$\wp_1$	1	1	1	1
$\wp_2$	t-1	-1	t-1	-1
$\wp_3$	s-1	s-1	-1	-1
$\wp_4$	α	1−s	1−t	1

Where , α=st−s−t−1 , $\mathcal{H}_1$={id} , $\mathcal{H}_2$={ s , 2s,……,(t−1)s} , $\mathcal{H}_3$={t ,2t,…..,(s-1)t} , $\mathcal{H}_4=\{a \in Z_{st} : (a, st) = 1\}$ are the rational conjugacy classes .

(ii) If s, r and t are distinct prime numbers such that $s > t > r$, then ($\equiv^*(Z_{str})$) =

QCT(Z_{str})	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_6$	$\mathcal{H}_7$	$\mathcal{H}_8$
$\wp_1$	1	1	1	1	1	1	1	1
$\wp_2$	r–1	-1	r-1	-1	r-1	-1	r-1	-1
$\wp_3$	t-1	t–1	-1	-1	t-1	t-1	-1	-1
$\wp_4$	α_1	1-t	1-r	1	α_1	1-t	1-r	1
$\wp_5$	s-1	s-1	s-1	s-1	-1	-1	-1	-1
$\wp_6$	α_2	1-s	α_2	1-s	1-r	1	1-r	1
$\wp_7$	α_3	α_3	1- s	1-s	1-t	1-t	1	1
$\wp_8$	α_4	-α_3	-α_2	s-1	-α_1	t-1	r-1	-1

Where , α_1 =(t-1)(r-1) , α_2=(s-1)(r-1) , α_3=(s-1)(t-1) , α_4=(s-1)(t-1)(r-1) and, the rational conjugacy classes are: $\mathcal{H}_1$={id}, $\mathcal{H}_2$={st , 2st ,…, (r-1)st} , $\mathcal{H}_3$={sr , 2sr ,…, (t-1)sr} , $\mathcal{H}_4$={tr, 2tr ,…., (s-1)tr} , $\mathcal{H}_5$={s, 2s ,… , (r– 1)(t–1)s} , $\mathcal{H}_6$={ t ,2t , … , (r-1)(s-1)t} , $\mathcal{H}_7$={r , 2r, …, (t-1)(s-1)r} and $\mathcal{H}_8=\{a \in Z_{str} : (a, str) = 1\}$.

3.The Cyclic Decomposition of The Rational Valued Character Table of Some Finite Groups:

In this section ,we will introduce some results about finding the cyclic. decomposition of the $D(\equiv^*(Z_{st}))$, $D(\equiv^*(Z_{str}))$, $D(\equiv^*(Z_{st} \times C_2))$ and $D(\equiv^*(Z_{str} \times C_2))$,where $s > t > r > 2$ are distinct prime numbers.

Theorem(3-1):

Let $\mathcal{K} = Z_{st}$ such that $|Z_{st}| = st$, where $s > t$ are distinct prime numbers, then the cyclic decomposition of the $\equiv^*(Z_{st})$ is $K(\equiv^*(Z_{st}))= Z_{st} \oplus Z_s \oplus Z_t \oplus Z$

Proof:

By theorem(2-9) we conclusion that $\equiv^*(Z_{st})$ is

$$\equiv^*(Z_{st})=\begin{bmatrix} 1 & 1 & 1 & 1 \\ t-1 & -1 & t-1 & -1 \\ s-1 & s-1 & -1 & -1 \\ \alpha & 1-s & 1-t & 1 \end{bmatrix}$$, where α=st −s−t +1

And, by using elementary rows and columns operations , we have the following

the diagonal matrix : $\begin{bmatrix} st & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

Thus , cyclic decomposition of $(\equiv^*(Z_{st}))$ is $K(\equiv^*(Z_{st}))= Z_{st} \oplus Z_s \oplus Z_t \oplus Z$

In the same way the theory below is proven

Theorem(3-2):

If $\mathcal{K} = Z_{str}$ is a finite group of order str , where $s > t > r$ are distinct prime numbers, then the cyclic decomposition of the $\equiv^*(Z_{str})$ is $K(\equiv^*(Z_{str}))= Z_{str} \oplus Z_{st} \oplus Z_{sr} \oplus Z_s \oplus Z_{tr} \oplus Z_t \oplus Z_r \oplus Z$.

Theorem (3-3):

Let $\mathcal{K} = Z_{st}$ such that $|Z_{st}|= st$, where $s > t > 2$ are distinct prime, then the rational character table of $(Z_{st} \times C_2)$ is $\equiv^*(Z_{st} \times C_2)=(\equiv^*(Z_{st}))\otimes(\equiv^*(C_2))$

Proof :

It is easy to see g. c. d ($|Z_{st}| \times |C_2|$)=g. c. d (st ,2)=1, so by theorem(2-8) we get:

$\equiv^*(Z_{st} \times C_2)= (\equiv^*(Z_{st})) \otimes (\equiv^*(C_2))$.

Where , $(\equiv^*(Z_{st}))\otimes (\equiv^*(C_2)) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t-1 & -1 & t-1 & -1 \\ s-1 & s-1 & -1 & -1 \\ \alpha & 1-s & 1-t & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ t-1 & t-1 & -1 & -1 & t-1 & t-1 & -1 & -1 \\ t-1 & 1-t & -1 & 1 & t-1 & 1-t & -1 & 1 \\ s-1 & s-1 & s-1 & s-1 & -1 & -1 & -1 & -1 \\ s-1 & 1-s & s-1 & 1-s & -1 & 1 & -1 & 1 \\ \alpha & \alpha & 1-s & 1-s & 1-t & 1-t & 1 & 1 \\ \alpha & -\alpha & 1-s & s-1 & 1-t & t-1 & 1 & -1 \end{bmatrix}_{8\times 8} , \alpha = st - s - t + 1$$

Theorem(3-4):

Let $\mathcal{K} = Z_{str}$ such that $|Z_{str}| = str$, where $s > t > r > 2$ are distinct prime numbers , then $\equiv^*(Z_{str} \times C_2) = (\equiv^*(Z_{str})) \otimes (\equiv^*(C_2))$.

Proof :

It is clearly that g. c. d ($|Z_{str}| \times |C_2|$)=g. c. d (str ,2)=1 and by theorem(2-8) we conclusion that :

$\equiv^*(Z_{str} \times C_2)= (\equiv^*(Z_{str})) \otimes (\equiv^*(C_2))$.

Where , $\equiv^*(Z_{str} \times C_2)= (\equiv^*(Z_{str})) \otimes (\equiv^*(C_2))$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ r-1 & -1 & r-1 & -1 & r-1 & -1 & r-1 & -1 \\ t-1 & t-1 & -1 & -1 & t-1 & t-1 & -1 & -1 \\ \alpha_1 & 1-t & 1-r & 1 & \alpha_1 & 1-t & 1-r & 1 \\ s-1 & s-1 & s-1 & s-1 & -1 & -1 & -1 & -1 \\ \alpha_2 & 1-s & \alpha_2 & 1-s & 1-r & 1 & 1-r & 1 \\ \alpha_3 & \alpha_3 & 1-s & 1-s & 1-t & 1-t & 1 & 1 \\ \alpha_4 & -\alpha_3 & -\alpha_2 & s-1 & -\alpha_1 & t-1 & r-1 & -1 \end{bmatrix}_{8\times 8} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$$

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$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
r-1 & r-1 & -1 & -1 & r-1 & r-1 & -1 & -1 & r-1 & r-1 & -1 & -1 & r-1 & r-1 & -1 & -1 \\
r-1 & 1-r & -1 & 1 & r-1 & 1-r & -1 & 1 & r-1 & 1-r & -1 & 1 & r-1 & 1-r & -1 & 1 \\
t-1 & t-1 & t-1 & t-1 & -1 & -1 & -1 & -1 & t-1 & t-1 & t-1 & t-1 & -1 & -1 & -1 & -1 \\
t-1 & 1-t & t-1 & 1-t & -1 & 1 & -1 & 1 & t-1 & 1-t & t-1 & 1-t & -1 & 1 & -1 & 1 \\
\alpha_1 & \alpha_1 & 1-t & 1-t & 1-r & 1-r & 1 & 1 & \alpha_1 & \alpha_1 & 1-t & 1-t & 1-r & 1-r & 1 & 1 \\
\alpha_1 & -\alpha_1 & 1-t & t-1 & 1-r & r-1 & 1 & -1 & \alpha_1 & -\alpha_1 & 1-t & t-1 & 1-r & r-1 & 1 & -1 \\
s-1 & s-1 & s-1 & s-1 & s-1 & s-1 & s-1 & s-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
s-1 & 1-s & s-1 & 1-s & s-1 & 1-s & s-1 & 1-s & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
\alpha_2 & \alpha_2 & 1-s & 1-s & \alpha_2 & \alpha_2 & 1-s & 1-s & 1-r & 1-r & 1 & 1 & 1-r & 1-r & 1 & 1 \\
\alpha_2 & -\alpha_2 & 1-s & s-1 & \alpha_2 & -\alpha_2 & 1-s & s-1 & 1-r & r-1 & 1 & -1 & 1-r & r-1 & 1 & -1 \\
\alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 & 1-s & 1-s & 1-s & 1-s & 1-t & 1-t & 1-t & 1-t & 1 & 1 & 1 & 1 \\
\alpha_3 & -\alpha_3 & \alpha_3 & -\alpha_3 & 1-s & s-1 & 1-s & s-1 & 1-t & t-1 & 1-t & t-1 & 1 & -1 & 1 & -1 \\
\alpha_4 & \alpha_4 & -\alpha_3 & -\alpha_3 & -\alpha_2 & -\alpha_2 & s-1 & s-1 & -\alpha_1 & -\alpha_1 & t-1 & t-1 & r-1 & r-1 & -1 & -1 \\
-\alpha_4 & \alpha_4 & -\alpha_3 & \alpha_3 & -\alpha_2 & \alpha_2 & s-1 & 1-s & -\alpha_1 & \alpha_1 & t-1 & 1-t & r-1 & 1-r & -1 & 1
\end{bmatrix}_{\mathbf{16\times16}}
$$

Where , α_1 =(t-1)(r-1) , α_2=(s-1)(r-1) , α_3=(s-1)(t-1) , α_4=(s-1)(t-1)(r-1).

Theorem (3-7):

If $\mathcal{K} = Z_{st}$ is group of order st , where $s > t > 2$ are distinct prime numbers , then K(≡*($Z_{st} \times C_2$))=K(≡*(Z_{st}))⊗K(≡*(C_2)).

Proof : From Theorem(3-1) ⟹D(≡*(Z_{st}))= Diag {st , s , t ,1} , and

Theorem(2-7) ⟹ D(≡*(C_2))=Diag{2,1} .

So , by using Theorem (3-4) and Theorem(2-5) we get:

D(≡*($Z_{pq} \times C_2$))=D(≡*(Z_{st}))⊗ D(≡*(C_2))=Diag{st , s , t ,1} ⊗ Diag{2 ,1}=

Diag{2st , 2s ,2t ,2,st , s , t , 1} .

Theorem(3-8): If $\mathcal{K}$ =Z_{str} is group of order str , where $s > t > r > 2$ are distinct prime numbers, then D(≡*($Z_{str} \times C_2$))= D((≡*(Z_{str})) ⊗ D((≡*(C_2)) .

Proof :

By Theorem(3-2) ⟹D(≡*(Z_{str}))= Diag{ str , st , sr ,s ,tr , t , r ,1}, and by

Theorem(2-7) ⟹D(≡*(C_2))=Diag{2,1}.

Therefore, by Theorem(3-5) and Theorem(2-5) we obtain:

D(≡*($Z_{str} \times C_2$))=D(≡*(Z_{str}))⊗D(≡*(C_2))=Diag{str, st , sr ,s ,tr , t , r ,1}⊗

Diag{2,1}= Diag{2str , 2st ,2sr ,2s ,2tr,2t ,2r, 2 , str, st , sr ,s, tr , t , r ,1} .

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