WEAK SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS OF PARTIAL DIFFERENTIAL EQUATIONS USING VARIATIONAL METHOD

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Abstract

This research delves into the examination of weak solutions for boundary value problems associated with nonlinear partial differential equations. Utilizing the variational method, we explore the conditions necessary and sufficient for the existence and uniqueness of these weak solutions. Furthermore, we provide practical demonstrations by solving specific examples of nonlinear problems involving partial differential equations. The study concludes with an analysis of the benefits and limitations inherent in employing the variational method for such investigations.

KEYWORDS: Weak Solutions, Nonlinear Partial Differential Equations, Boundary Value Problems, Variational Method, Existence, Uniqueness, Criteria, Examples.

1. INTRODUCTION

The study of nonlinear partial differential equations (PDEs) encompasses a wide array of physical phenomena and mathematical challenges. The pursuit of weak solutions for boundary value problems in this context is of paramount importance, as it sheds light on the existence and uniqueness of solutions using the variational method. In this paper, we embark on an exploration of the variational approach to ascertain the criteria for the existence and uniqueness of weak solutions. Variational methods have been widely employed to study nonlinear PDEs due to their ability to provide insights into the existence and uniqueness of solutions. The work of [1] extensively explores variational techniques in the context of PDEs, highlighting their versatility in handling a broad class of nonlinear problems. The connection between weak solutions and variational methods has been a focal point in research. [2] presents a comprehensive overview of weak solutions and their relation to variational calculus, emphasizing their significance in nonlinear PDEs. The application of variational methods in solving nonlinear PDEs is well-documented in [3]. Also, [3] discuss the applications of variational techniques in the study of nonlinear elliptic and parabolic equations, providing valuable insights into their efficacy. Understanding the benefits and limitations of variational methods is crucial for researchers in the field. The work of [4] offers a comprehensive perspective on the advantages and challenges associated with variational techniques in nonlinear PDEs. This research also aims to solve specific nonlinear problems involving PDEs, thereby illustrating the practical application of the variational method.

2. PRELIMINARY

Definition (*Weak Solution) 2.1.* A weak solution (also referred to as generalized solution) to an ordinary differential equation or partial differential equation is a function for which the derivatives may not all exist but which is none the less deemed to satisfy the equation in some precisely defined sense.

Definition (Variational Method) 2.2. The variational methods for boundary value problem requires notions and properties of function spaces, and notions, properties from operator theory and their applications in variational principle. Let's look at the treatment of definitions, notions and results from function spaces, operator theory and variational principle.

example of the equation

$$h(x) = 0 \qquad (2.2.0)$$

where$h(x)$is a continuous real function on $\mathbb{R}$. The problem of determining the local extremum $(\text{e.g.}, \text{the minimum})$ of the function H which is a primitive of $h,$ i.e., for which

$$H^{'}(x) = h(x) \qquad (2.2.1)$$

is valid for every $x \in \mathbb{R}$. In essence, if a point $x_0 \in \mathbb{R}$ exists at which the function $H(x)$ assumes its local minimum, then the derivative $H^{'}(x)$ necessarily vanishes by familiar theorems of classical analysis and x_0 is thus the solution of equation $(2.2.0)$. This is just the idea which we will wish to exploit to ensure the existence of the solution of the operator equation

$$Au = f \qquad (2.2.2)$$

In particular, we will be interested in the case when A is a differential operatorand$X = V$, where $V$is the reflexive Banach space. The expression

$$Au - f$$

will take the role of the function h in equation $(2.2.0)$; however, when constructing the 'primitive' H the following problem is immediately encountered: what really is the 'derivative' in the case?

Equation $(2.2.2)$ is solved in the space X. Therefore, we first develop a theory of differentiation for a functional, defined on normed linear space. The problem is then that of associating, with the operator $A,$ a functional F defined on Xin such a way that its 'derivative' is the expression

$$Au - f.$$

We will try to prove that the functional F has a minimum on the space X. If the analogue of the corresponding theorem from calculus is still valid (i.e if it holds that the derivative of the functional F vanishes at the point of the minimum), this will also serve as the proof of the existence of the weak solution of boundary value problem for the formal differential operator $Au - f$.

Definition (Nonlinear Partial Differential Equations) 2.3. A partial differential equation $\mathrm{F}(X, U, \ldots, D^{\alpha}u) = 0$ is called nonlinear partial differential equation if the function $\mathrm{F}(X, U, \ldots, D^{\alpha}u)$ is not linear in any of the arguments $(U, \ldots, D^{\alpha}u)$.

3. SOLUTIONS

The central idea revolves around the exploration of weak solutions for boundary value problems associated with nonlinear partial differential equations using the variational method. This involves the establishment of necessary and sufficient criteria for the existence and uniqueness of weak solutions. We shall consider specific examples of nonlinear partial differential equations to check if boundary value problems have at least one weak solution

Example 3.1. Show that the boundary value problem

$$-\Delta u + u|u|^{p-2} = f \ on \ \Omega, u = \phi \ on \ \partial\Omega.$$

has at least one weak solution.

Solution

Here, we need to show that the boundary value problem stated above has at least one weak solution. Therefore, recall the general form of partial differential equation of order $2k$

$$\sum_{|\alpha|\leq k} (-1)^{|\alpha|} D^{\alpha} a_{\alpha}\big(x, \delta_k u(x)\big) = f(x) \qquad (3.1.0)$$

We now seek to obtain the coefficient of a_α. From the example, we have

$$-\Delta u + u|u|^{p-2} = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}\right) + u|u|^{p-2}$$

$$= \quad -\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)\right) + u|u|^{p-2}$$

$$= \sum_{i=1}^{N} (-1)^i \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) + u|u|^{p-2}$$

$$= \sum_{i=1}^{N} (-1)^i \frac{\partial}{\partial x_i}(\eta_i) + \eta_0^{p-2}\eta_0$$

$$= \sum_{i=1}^{N} (-1)^i \frac{\partial}{\partial x_i}(\eta_i) + \eta_0^{p-2+1}$$

$$\Rightarrow \sum_{i=1}^{N} (-1)^i \frac{\partial}{\partial x_i}(\eta_i) + |\eta_0|^{p-1} = f(x) \qquad (3.1.1)$$

Comparing $(3.1.1)$ with $(3.1.0)$, we have

$$a_i = \eta_i \ \ and \ a_0 = |\eta_0|^{p-1}$$

We now investigate the equation above to see if there exist at least one weak solution by verifying that the conditions of theorem on existence of weak solution in chapter three are satisfied by the coefficients $a_i = \eta_i \ \ and \ a_0 = |\eta_0|^{p-1}$.

To show that $a_\alpha(x,\eta) \in CAR^*(p)$, we show that the coefficient a_α satisfy the growth conditions below

$$|a_i| = \eta_i \leq \sum_{i=0}^{N} |\eta_i|^{p-1} \leq c_i \left[g_i(x) + |\eta_0|^{p-1} + \sum_{i=1}^{N} |\eta_i|^{p-1} \right],$$

And$|a_0| = |\eta_0|^{p-1} \leq \sum_{i=0}^{N} |\eta_i|^{p-1} \leq c_i[g_i(x) + |\eta_0|^{p-1} + \sum_{i=0}^{N} |\eta_i|^{p-1}]$

where$c_1 = 1, g_i(x) = 0, i = 1,2, \dots, N, 1 < p < N$.

For $p > N$, we have

$$|a_i| = \eta_i \leq \sum_{i=0}^{N} |\eta_i|^{p-1} \leq c_i(|\eta_0|) \left[g_i(x) + \sum_{i=0}^{N} |\eta_i|^{p-1} \right],$$

and

$$|a_0| = |\eta_0|^{p-1} \leq \sum_{i=1}^{N} |\eta_0|^{p-1} \leq c_i(|\eta_0|) \left[g_i(x) + \sum_{i=0}^{N} |\eta_i|^{p-1} \right]$$

Where$c_i(|\eta_0|) = 1, g_i(x) = 0, i = 1,2, \dots, N, p > 1, p > N$.

This shows that the coefficients $a_i = \eta_i$and $a_0 = |\eta_0|^{p-1}$ satisfy $a_i(x,\eta) \in CAR^*(p)$ for all $p > 1$ and so $a_\alpha(x,\eta) \in CAR^*(p)$ is verified.

We now show that the potentiality condition is satisfied. That is

$a_{\alpha\beta}(x,\eta) = a_{\beta\alpha}(x,\eta)and\eta_\alpha a_{\alpha\beta}(x,\eta) \in CAR^*(p)$ hold,

Now

$$a_{ij}(x,\eta) = \frac{\partial a_i}{\partial \eta_j} = \frac{\partial a_0}{\partial \eta_i} = \frac{\partial |\eta_0|^{p-1}}{\partial \eta_i} = 0.$$

$$a_{ji}(x,\eta) = \frac{\partial a_i}{\partial \eta_0} = \frac{\partial \eta_i}{\partial \eta_j} = 0$$

Therefore $a_{ij}(x,\eta) = a_{ji}(x,\eta) = 0$

Since $a_{ij}(x,\eta) = 0$, then

$$\eta_i a_{ij}(x,\eta) = 0$$

$$\Longrightarrow \left|\eta_i a_{ij}(x,\eta)\right| = |0| \le \sum_{i=0}^{N} |\eta_i|^{p-1} \le c_i \left[g_i(x) + |\eta_0|^{p-1} + \sum_{i=0}^{N} |\eta_i|^{p-1} \right]$$

where $i = 1,2,\ldots,N, c_i = 1, g_i(x) = 0, p > 1$. That is

$a_{ij}(x,\eta) = a_{ji}(x,\eta)and\eta_i a_{ij}(x,\eta) \in CAR^*(p)$ are satisfied

We now show that the monotonicity condition is satisfied. That is

$$\sum_{|\alpha|\le k} [a_\alpha(x,\eta) - a_\alpha(x,\gamma)]\,(\eta_\alpha - \gamma_\alpha) \ge 0$$

holds for all $\eta, \gamma \in \mathbb{R}^m$, now

$$\sum_{|\alpha|\le k} [a_\alpha(x,\eta) - a_\alpha(x,\gamma)](\eta_\alpha - \gamma_\alpha)$$

$$= \sum_{i=1}^{N} [a_i(x,\eta) - a_i(x,\gamma)](\eta_i - \gamma_i) + \big(a_0(x,\eta) - a_0(x,\gamma)\big)(\eta_0 - \gamma_0)$$

$$= \sum_{i=1}^{N} (\eta_i - \gamma_i)(\eta_i - \gamma_i) + (|\eta_0|^{p-1} - |\gamma_0|^{p-1})(\eta_0 - \gamma_0)$$

$$\sum_{i=1}^{N}(\eta_i-\gamma_i)(\eta_i-\gamma_i)+(|\eta_0|^{p-1}-|\gamma_0|^{p-1})(\eta_0-\gamma_0)\geq 0$$

If $\eta_0 \geq \gamma_0$ or $\gamma_0 \leq \eta_0$ then

$$\sum_{i=1}^{N}(\eta_i-\gamma_i)(\eta_i-\gamma_i)+(|\eta_0|^{p-1}-|\gamma_0|^{p-1})(\eta_0-\gamma_0)\geq 0$$

Therefore, the monotonicity condition above is satisfied

Finally, we show that the coercivity condition holds.

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha \geq c_1\sum_{|\alpha|=k}|\eta_\alpha|^p + c_2|\eta_0|^p - c_3 \qquad (3.1.2)$$

But

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha = \sum_{i=1}^{N}(\eta_i)\,\eta_i + |\eta_0|^{p-1}\eta_0$$

$$=\sum_{i=1}^{N}\eta_i^2+\eta_0^{p-1}\eta_0$$

$$=\sum_{i=1}^{N}\eta_i^2+\eta_0^{p-1+1}$$

$$=\sum_{i=1}^{N}\eta_i^2+\eta_0^{p}$$

Put

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha = \sum_{i=1}^{N}\eta_i^2+\eta_0^{p}$$

in equation (3.1.2)

$$\sum_{i=1}^{N} \eta_i^2 + \eta_0^p \geq c_1 \sum_{|\alpha|=k} |\eta_\alpha|^p + c_2|\eta_0|^p - c_3 \quad \Rightarrow$$

$$\sum_{i=1}^{N} \eta_i^2 - c_1 \sum_{i-1}^{N} |\eta_i|^p + \eta_0^p - c_2|\eta_0|^p + c_3 \geq 0 \ \Rightarrow$$

$$\sum_{i=1}^{N} \eta_i^2 - c_1 \sum_{i-1}^{N} |\eta_i|^p + \eta_0^p(1 - c_2) + c_3 \geq 0 \qquad (3.1.3)$$

We choose

$$c_1 = \left(\sum_{i-1}^{N} |\eta_i|^p \right)^{-1}, c_2 = 1, c_3 = 2$$

then by substitution into (3.1.3) we have

$$\sum_{i=1}^{N} \eta_i^2 - \left(\sum_{i-1}^{N} |\eta_i|^p \right)^{-1} \times \sum_{i-1}^{N} |\eta_i|^p + \eta_0^p(1-1) + 2 \geq 0 \ \Rightarrow$$

$$\sum_{i=1}^{N} \eta_i^2 - 1 + 2 \geq 0 \ \Rightarrow$$

$$\sum_{i=1}^{N} \eta_i^2 + 1 \geq 0$$

Therefore, the coercivity condition is satisfied and the boundary value problem under consideration has at least one weak solution.

Example 3.2. Verify that the Dirichlet problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f(x) \ on \ \Omega, u = \phi \ on \ \partial\Omega$$

has at least one weak solution.

Solution

We first find the coefficient of a_α, as follows

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)=f(x)$$

$$=\sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}\left(\left(\frac{\partial u}{\partial x_i}\right)^{p-2}\frac{\partial u}{\partial x_i}\right)$$

$$=\sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{p-2+1}$$

$$=\sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{p-1}$$

$$\Rightarrow\sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{p-1}=f(x)$$

Comparing this with $(3.1.0)$, we have $a_i=|\eta_i|^{p-1}\ and\ a_0=0$. We now want to investigate to see whether or not ***Example 3.2*** has at least one weak solution by verifying the conditions

We first verify the growth condition

$$|a_i|=|\eta_i|^{p-1}\leq\sum_{i=0}^{N}|\eta_i|^{p-1}$$

$$\leq c_1\left[g_i(x)+|\eta_0|^{p-1}+\sum_{i=1}^{N}|\eta_i|^{p-1}\right],$$

where $c_1=1, g_i(x)=0, i=1,2,\dots,N, 1<p<N$.

For $p>N$, we have

$$|a_i|=|\eta_i|^{p-1}\leq\sum_{i=0}^{N}|\eta_i|^{p-1}$$

$$\leq c_1(|\eta_0|)\left[g_i(x)+\sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

Where $c_1(|\eta_0|)=1, g_i(x)=0, i=1,2,\ldots,N, p>1, p>N$.

This shows that the coefficients $a_i=|\eta_i|^{p-1}$ satisfy $a_i \in CAR^*(p)$ for all $p>1$ and so $a_\alpha(x,\eta)\in CAR^*(p)$ is verified.

We now show that the potentiality condition is satisfied. That is

$a_{\alpha\beta}(x,\eta)=a_{\beta\alpha}(x,\eta)and\eta_\alpha a_{\alpha\beta}(x,\eta)\in CAR^*(p)$ hold

Now

$$a_{ij}(x,\eta)=\frac{\partial a_i}{\partial \eta_j}=\frac{\partial a_0}{\eta_i}=0=\frac{\partial|\eta_i|^{p-1}}{\partial \eta_0}=\frac{\partial a_i}{\eta_0}=\frac{\partial a_j}{\partial \eta_i}=a_{ji}(x,\eta)$$

and

$$\left|\eta_i a_{ij}(x,\eta)\right|=|0|\leq\sum_{i=1}^{N}|\eta_i|^{p-1}\leq c_i\left[g_i(x)+|\eta_0|^{p-1}+\sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

where $i=1,2,\ldots,N, c_1=1, g_i(x)=0, p>1$. That is $a_{ij}(x,\eta)=a_{ji}(x,\eta)$ and $\eta_i a_{ij}(x,\eta)\in CAR^*(p)$ are satisfied.

We now show that the monotonicity condition is satisfied, that is

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

holds for all $\eta,\gamma\in\mathbb{R}^m$. Now

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)$$

$$= \sum_{i=1}^{N} [a_i(x,\eta) - a_i(x,\gamma)](\eta_i - \gamma_i) + \big(a_0(x,\eta) - a_0(x,\gamma)\big)(\eta_0 - \gamma_0)$$

$$= \sum_{i=1}^{N} (|\eta_i|^{p-1} - |\gamma_i|^{p-1})(\eta_i - \gamma_i) + 0 + 0(\eta_0 - \gamma_0)$$

$$= \sum_{i=1}^{N} (|\eta_i|^{p-1} - |\gamma_i|^{p-1})(\eta_i - \gamma_i) \geq 0$$

Now if $\eta_i > \gamma_i$, then $|\eta_i|^{p-1} > |\gamma_i|^{p-1}$ and so $\eta_i - \gamma_i > 0$. Therefore the monotonicity condition holds.

Finally we show that $(3.1.2)$holds.

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha \geq c_1 \sum_{|\alpha|=k} |\eta_\alpha|^p + c_2|\eta_0|^p - c_3 \qquad (3.1.2)$$

But

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha = \sum_{i=1}^{N} |\eta_i|^{p-1}\eta_i = \sum_{i=1}^{N} |\eta_i|^p$$

Therefore from $(3.1.2)$, we have

$$\sum_{i=1}^{N} |\eta_i|^p \geq c_1 \sum_{|\alpha|=k} |\eta_\alpha|^p + c_2|\eta_0|^p - c_3$$

$$\sum_{i=1}^{N} |\eta_i|^p - c_1 \sum_{i=1}^{N} |\eta_i|^p - c_2|\eta_0|^p + c_3 \geq 0$$

Choose

$$c_1 = \left(\sum_{i=1}^{N} |\eta_i|^p\right)^{-1}, c_2 = 0, c_3 = 2$$

We have by substitution

$$\sum_{i=1}^{N}|\eta_i|^p + 1 \geq 0$$

Therefore $(3.1.2)$ holds and the boundary value problem has at least one weak solution.

Example 3.3. Consider the formal differential operator

$-\Delta u + |u|^s u = Au$ on Ω

where$s$ is a positive parameter. Show that the problem above has at least one weak solution in the $W^{1,2}(\Omega)$ under Dirichlet boundary value condition.

For $s \geq 0$ when $N = 2$ and $s \leq \frac{4}{s-2}$ when $N > 2$, we have $p \geq s + 1$

Solution

$$-\Delta u + |u|^s u = Au$$

$$-\Delta u + |u|^s u = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) + |u|^s u$$

$$= \sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) + u^{s+1}$$

$$= \sum_{i=1}^{N}(-1)^i\frac{\partial}{\partial x_i}(\eta_i) + |\eta_0|^{s+1} \qquad (3.3.0)$$

Comparing $(3.3.0)$with $(3.1.0)$, we have $a_i = \eta_i$ and $a_0 = |\eta_0|^{s+1}$

We now show that $a_\alpha \in CAR^*(p)$ by verifying the growth conditions.

Since $p = 2 \text{ and } \mathrm{N} \geq 2$, then we cannot have the case of $p > N$.

For $p \leq N$

$$|a_i| = |\eta_i| \leq \sum_{i=1}^{N}|\eta_i|^{p-1}$$

$$\leq c_1\left[g_i(x)+|\eta_0|^{p-1}+\sum_{i=1}^{N}|\eta_i|^{p-1}\right],$$

Where$i=1,2,\dots,N$.

$$|a_0|=|\eta_0|^{s+1}\leq\sum_{i=1}^{N}|\eta_0|^{p-1}\leq c_1\left[g_i(x)+|\eta_0|^{p-1}+\sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

Where$c_1=1, g_i(x)=0, i=1,2,\dots,N$.

and

$$|a_0|=|\eta_0|^{s+1}\leq|\eta_0|^{p}\leq c_0\left[g_0(x)+|\eta_0|^{p-1}+\sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

$$c_0=1, g_0(x)=1, p\geq s+1, p\geq 0$$

This shows that $a_\alpha\in CAR^*(p), i=0,1,2,\dots,N,$ for $p\leq N$

We now show that the potentiality condition is satisfied. That is

$a_{\alpha\beta}(x,\eta)=a_{\beta\alpha}(x,\eta)and\eta_\alpha a_{\alpha\beta}(x,\eta)\in CAR^*(p)$

$$a_{ij}(x,\eta)=\frac{\partial a_i}{\partial\eta_j}=\frac{\partial a_0}{\partial\eta_i}=\frac{\partial|\eta_0|^{s+1}}{\partial\eta_i}=0$$

$a_{ji}(x,\eta)=\frac{\partial a_j}{\partial\eta_i}=\frac{\partial a_i}{\partial\eta_0}=\frac{\partial\eta_i}{\partial\eta_0}=0$, therefore $a_{ij}(x,\eta)=a_{ji}(x,\eta)$ and we also show that

$\eta_i a_{ij}(x,\eta)\in CAR^*(p)$as follows

$$\left|\eta_i a_{ij}(x,\eta)\right|=|0|\leq c_1\left[g_i(x)+|\eta_i|^{p}+\sum_{i=0}^{N}|\eta_i|^{p}\right]$$

Where$c_1=1, g_i(x)=0, i=1,2,\dots,N, p\geq 0, p>2, p\leq N$

we now show that the monotonicity condition holds

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

holds for all $\eta,\gamma\in\mathbb{R}^m$. Now

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)$$

$$=\sum_{i=1}^{N}[a_i(x,\eta)-a_i(x,\gamma)](\eta_i-\gamma_i)+\big(a_0(x,\eta)-a_0(x,\gamma)\big)(\eta_0-\gamma_0)$$

$$=\sum_{i=1}^{N}(\eta_i-\gamma_i)(\eta_i-\gamma_i)+(|\eta_i|^{s+1}+|\gamma_i|^{s+1})(\eta_0-\gamma_0)$$

$$=\sum_{i=1}^{N}(\eta_i-\gamma_i)^2+(|\eta_i|^{s+1}+|\gamma_i|^{s+1})(\eta_0-\gamma_0)\geq 0$$

Now if $(\eta_i-\gamma_i)^2>0$, then $|\eta_i|^{s+1}>|\gamma_i|^{s+1}$and so $\eta_0-\gamma_0>0$, therefore the monotonicity condition holds.

Hence the expression

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

Next we show the coercivity condition holds

$$\sum_{|\alpha|\leq k}a_\alpha(x,\eta)\,\eta_\alpha\geq c_1\sum_{|\alpha|=k}|\eta_\alpha|^p+c_2|\eta_0|^p-c_3$$

But

$$\sum_{|\alpha|\leq k}a_\alpha(x,\eta)\,\eta_\alpha=\sum_{i=1}^{N}(\eta_i)\eta_i+|\eta_0|^{s+1}\eta_0$$

$$= \sum_{i=1}^{N} (\eta_i)^2 + \eta_0^{s+2}$$

By substitution, we have

$$\sum_{i=1}^{N} (\eta_i)^2 + \eta_0^{s+2} \geq c_1 \sum_{|\alpha|=k} |\eta_\alpha|^p + c_2 |\eta_0|^p - c_3$$

$$\sum_{i=1}^{N} (\eta_i)^2 + \eta_0^{s+2} - c_1 \sum_{i=1}^{N} |\eta_i|^p - c_2 |\eta_0|^p + c_3 \geq 0$$

Choose

$$c_1 = \left(\sum_{i=1}^{N} |\eta_i|^p \right)^{-1}, c_2 = (|\eta_0|^p)^{-1}, c_3 = 2$$

$$\sum_{i=1}^{N} (\eta_i)^2 + \eta_0^{s+2} - 1 - 1 + 2 \geq 0$$

$$\Rightarrow \sum_{i=1}^{N} (\eta_i)^2 + \eta_0^{s+2} \geq 0$$

Then

$$\sum_{|\alpha| \leq k} a_\alpha(x, \eta)\, \eta_\alpha \geq 0$$

Since $(\eta_i)^2 \geq 0$ and $\eta_0^{s+2} \geq 0$.

Therefore, there exist at least one weak solution of the boundary value problem.

Example 3.4. Let $h(\eta)$ be a continuous non decreasing function on $\mathbb{R}, f \in L_2(\Omega)$ and suppose that $c > 0$ exist such that $|g(\eta)| \leq c(|\eta|^\tau +)$ holds for all $\eta \in \mathbb{R}(\eta \subset \mathbb{R}^N, N > 1, \eta \in c^{0,1})$, where τ is an arbitrary positive number for $N = 2$, we have

$\tau = \frac{N+2}{N-2}$.

Prove that the Dirichelet problem for the formal differential equation

$$-\Delta u(x) + h\big(u(x)\big) = f(x) \text{ } on \text{ } \Omega$$

has at least one weak solution.

Solution

We first determine the coefficients of a_α as follows

$$-\Delta u(x) + h\big(u(x)\big) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) + h\big(u(x)\big) = \sum_{i=1}^{N} (-1)^i \frac{\partial}{\partial x_i}(\eta_i) + h(\eta_0) \qquad (3.4.0)$$

Comparing $(3.4.0)$ this with $(3.1.0)$, we have $a_i = \eta_i \text{ } and \text{ } a_0 = h(\eta_0)$

We now check to see whether example 4 with the coefficient above satisfy the boundedness property. That is to show that$a_\alpha \in CAR^*(p)$.

$$|a_i| = |\eta_i| \leq \sum_{i=1}^{N} |\eta_i|^p \leq c_i \left[g_i(x) + |\eta_0|^p + \sum_{i=i}^{N} |\eta_i|^{p-1} \right]$$

Where$i = 1,2, \dots, N$.

$$|a_0| \leq c_i(|\eta_0|) \left[g_i(x) + \sum_{i=1}^{N} |\eta_i|^{p-1} \right]$$

Where$c_i = 1, \text{ } g_i(x) = 0, \text{ } i = 1,2, \dots, N$.

For $p > N$

$$|a_i| \leq c_i(|\eta_0|) \left[g_i(x) + \sum_{i=1}^{N} |\eta_i|^{p-1} \right]$$

From the coefficients obtained above, for $p > N$

$$|a_i| = |\eta_i| \leq |\eta_0|^{p-1} \leq \sum_{i=1}^{N} |\eta_0|^{p-1} \leq c_i \left[g_i(x) + |\eta_0|^p + \sum_{i=1}^{N} |\eta_i|^{p-1} \right]$$

$$c_i = 1, g_i(x) = 0, i = 1,2, \dots, N$$

$$|a_0| = |h(\eta_0)| \leq c_0(|\eta_0|^{\tau} + 1) \leq c_0\left[|\eta_0|^{\tau} + 1 + \sum_{i=1}^{N}|\eta_i|^p\right] \leq c_0\left[g_0(x) + |\eta_i|^{\tau} + \sum_{i=1}^{N}|\eta_i|^p\right]$$

where $c_0 > 0, g_0(x) \equiv 1, \tau = \frac{N+2}{N-2}, p \geq 0$.

For $p > N$

$$|a_i| = |\eta_i| \leq \sum_{i=1}^{N}|\eta_i|^{p-1} \leq c_i(|\eta_0|)\left[g_i(x) + |\eta_0|^p + \sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

where $c_i = 1, g_i(x) \equiv 0$

$$|a_0| = |h(\eta_0)| \leq c_0|\eta_0|\left[g_0(x) + |\eta_0|^p + \sum_{i=1}^{N}|\eta_i|^p\right]$$

where $c_0 = |h(\eta_0)| + |h(-t)|\ and\ g_0(x) \equiv 1$

This shows that $a_\alpha \in CAR^*(p), i = 0,1,2,\dots,N$.

Therefore, the boundedness property is satisfied

We now show that the potentiality condition

$a_{\alpha\beta}(x,\eta) = a_{\beta\alpha}(x,\eta)$ and $\eta_\alpha a_{\alpha\beta}(x,\eta) \in CAR^*(p)$ are satisfied

$a_{ij} = \frac{\partial a_i}{\partial \eta_j} = \frac{\partial a_0}{\partial \eta_i} = \frac{\partial(h(\eta_0))}{\partial \eta_i} = 0$ and $a_{ji} = \frac{\partial a_j}{\partial \eta_i} = \frac{\partial a_i}{\partial \eta_0} = \frac{\partial \eta_i}{\partial \eta_0} = 0$

Therefore $a_{ij}(x,\eta) = a_{ji}(x,\eta) = 0, \text{since}\ a_{ij}(x,\eta) = 0$, then

$$\eta_i a_{ij}(x,\eta) = 0 \Longrightarrow \left|\eta_i a_{ij}(x,\eta)\right| = |0| \leq \sum_{i=1}^{N}|\eta_i| \leq c_i(|\eta_0|)\left[g_i(x) + \sum_{i=1}^{N}|\eta_i|^{p-1}\right]$$

where $i = 0,1,2,\dots,N, c_i = 1, g_i(x) = 0$.

Hence, the potentiality condition is satisfied.

Next we show that the monotonicity condition holds. That is

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

holds for all $\eta,\gamma\in\mathbb{R}^m$. But

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)$$

$$=\sum_{i=1}^{N}[a_i(x,\eta)-a_i(x,\gamma)](\eta_i-\gamma_i)+\big(a_0(x,\eta)-a_0(x,\gamma)\big)(\eta_0-\gamma_0)$$

$$=\sum_{i=1}^{N}(\eta_i-\gamma_i)(\eta_i-\gamma_i)+\big(h(\eta_0)-h(\gamma_0)\big)(\eta_0-\gamma_0)$$

$$=\sum_{i=1}^{N}(\eta_i-\gamma_i)^2+\big(h(\eta_0)-h(\gamma_0)\big)(\eta_0-\gamma_0).$$

then

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

For$\eta_i>\gamma_i$ for $i=0,1,2,\dots,N$ and also $\eta_i<\gamma_i$ since $h(\eta)$ is a non-decreasing function.

Similarly if $\eta_i=\gamma_i$ for $i=0,1,2,\dots,N$, then the expression above is equal to zero.

Therefore

$$\sum_{|\alpha|\leq k}[a_\alpha(x,\eta)-a_\alpha(x,\gamma)]\,(\eta_\alpha-\gamma_\alpha)\geq 0$$

and the monotonicity condition is verified.

Finally, we show that the coercivity condition holds, that is

$$\sum_{|\alpha|\leq k}a_\alpha(x,\eta)\,\eta_\alpha\geq c_1\sum_{|\alpha|=k}|\eta_\alpha|^p+c_2|\eta_0|^p-c_3$$

But

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta)\,\eta_\alpha = \sum_{|\alpha|=k} a_\alpha(x,\eta)\,\eta_\alpha = \sum_{i=1}^{N}(\eta_i)\eta_i + h(\eta_0)\eta_0$$

$$= \sum_{i=1}^{N}(\eta_i)^2 + h(\eta_0)\eta_0$$

We have by substitution

$$\sum_{i=1}^{N}(\eta_i)^2 + h(\eta_0)\eta_0 - c_1\sum_{i=1}^{N}|\eta_i|^p - c_2|\eta_0|^p + c_3 \geq 0$$

Choose

$$c_1 = \left(\sum_{i=1}^{N}|\eta_i|^p\right)^{-1}, c_2 = 0, c_3 = 1, \qquad \text{since the problem is Dirichelet type}$$

By substitution, we have

$$\sum_{i=1}^{N}(\eta_i)^2 + h(\eta_0)\eta_0 - 1 - 0 + 1 \geq 0 \Longrightarrow \sum_{i=1}^{N}(\eta_i)^2 + h(\eta_0)\eta_0 \geq 0$$

Hence the coercive condition is satisfied

Therefore, the boundary value problem has at least one weak solution.

4. CONCLUSION

In conclusion, this research contributes to the understanding of weak solutions in the realm of nonlinear partial differential equations. The variational method proves instrumental in discerning the conditions for the existence and uniqueness of such solutions. Through the application of this method to specific examples, we gain valuable insights into practical problem-solving. The study concludes with an assessment of the benefits and limitations inherent in the variational approach for investigating nonlinear boundary value problems in partial differential equations.

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