Robert Orthogonality in Normed Linear Spaces Via 2-HH Norm

Bhuwan Prasad Ojha, Prakash Muni Bajracharya

Abstract

The p-HH norms on X^2 were introduced by Kikianty and Dragomir in 2008. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Robert orthogonality in normed spaces and discuss some properties of this orthogonality.

Keywords: *Robert orthogonality, p-HH norm, Isosceles orthogonality, Pythagorean orthogonality, Hermite-Hadamards inequality* .

1 Introduction

The p-HH norms are equivalent to p-norms on X^2, as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:

Definition. [6, 9] For any convex function $f : [a, b] \to \mathbb{R}([a, b] \subset \mathbb{R}$, the Hermite-Hadamard's inequality is defined as

$$(b-a)f(\frac{a+b}{2}) \leq \int_a^b f(t)dt \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

. This inequality has been extended (see-12) for convex function $f : [x, y] \to \mathbb{R}$, where $[x, y] = \{(1-t)x + ty, t \in [0, 1]\}$. In that case Hermite-Hadamards integral inequality becomes

$$f(\frac{x+y}{2)} \leq \int_0^1 f\left[(1-t)x + ty\right]dt \leq \frac{f(x)+f(y)}{2} \qquad ......(1).$$

Using the convexity of $f(x) = \|x\|^p \quad (x \in X, p \geq 1)$ and relation (1) we have

$$\left\|\frac{x+y}{2}\right\| \leq \left[\int_0^1 \|(1-t)x + ty\|^p \, dt\right]^{\frac{1}{p}} \leq \frac{1}{2^{\frac{1}{p}}}(\|x\|^p + \|y\|^p)^{\frac{1}{p}}.$$

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[1]Corresponding author: Bhuwan Prasad Ojha(PhD student, Tribhuvan University, kathmandu, Nepal)
email: bhuwanp.ojha@apexcollege.edu.np
Co-author: Dr. Prakash Muni Bajracharya(Professor, Tribhuvan University, kathmandu, Nepal)
email: pmbajracharya13@gmail.com

Definition. (Generalized Geometric Mean)[9] The generalized geometric mean of order p of x and y ($x, y > 0$,p is an extended real number) is defined by

$$L^{[p]}(x,y) = \begin{cases} [\frac{1}{p+1}(\frac{y^{p+1}-x^{p+1}}{y-x}]^{\frac{1}{p}}, & \text{if} \quad p \neq -1, 0, \pm\infty \\ \frac{y-x}{logy-logx}, & \text{if} \quad p = -1 \\ \frac{1}{e}(\frac{y^y}{x^x})^{\frac{1}{1-y}}, & \text{if} \quad p = 0 \\ max\{x,y\} & \text{if} \quad p = \infty \\ min\{x,y\} & \text{if} \quad p = -\infty \end{cases} \quad \text{and} \quad L^{[p]}(x,x) = x$$

A function $d : X \times X \to \mathbb{R}$ is called a metric on X if it satisfies the following conditions:

1. $\forall x, y \in X, \quad d(x,y) \geq 0.$
2. $\forall x, y \in X, \quad d(x,y) = 0 \Leftrightarrow x = y.$
3. $\forall x, y \in X, \quad d(x,y) = d(y,x).$
4. $\forall x, y, z \in X, \quad d(x,y) \leq d(x,z) + d(z,y)$

[1] A vector space X is said to be normed space if there is a mapping $\|.\| : X \to \mathbb{R}$ on X satisfying the properties:

1. $\|x\| \geq 0, \quad \|x\| = 0 \Leftrightarrow x = 0.$
2. $\|\lambda x\| = |\lambda| \, \|x\|.$
3. $\|x+y\| \leq \|x\| + \|y\|.$

An inner-product space on a vector space X is a mapping $\langle ., . \rangle : x \times \to \mathbb{K}$, where ($\mathbb{K} = \mathbb{R} or \mathbb{C}$ satisfying the following properties:

1. $\forall x, y, z \in X, \quad \langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$
2. $\forall x, y \in X$ and$\alpha \in \mathbb{K}, \quad \langle \alpha x, \rangle = \alpha \langle x, y \rangle.$
3. $\forall x, y \in X, \quad \langle x, y \rangle = \langle y, x \rangle.$
4. $\forall x \in X, \quad \langle x, x \rangle \geq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0.$

An inner-product on X defines a norm on X by $\|x\|^2 = \langle x, x \rangle$. Every innerproduct spaces are normed spaces, but the converse may not be true. A best example of normed space which is not an inner-product space is $l^p = \{(x_n), x_n \in \mathbb{R} : \sum |x_n| < \infty\}$ for $p \neq 2$.

Definition. [7] The $p - HH$ norm on $X^2 = X \times X$ is defined by

$$\|(x,y)\|_{p-HH} = (\int_0^1 \|(1-t)x + ty\|^p \, dt)^{\frac{1}{p}}$$

for any $x, y \in X^2$ and $1 \leq p < \infty$.

The 2-HH norm is defined as follows:

$$\begin{aligned} \|(x,y)\|^2_{2-HH} &= \int_0^1 \|(1-t)x + ty\|^2 \, dt \\ &= \frac{1}{3}[\|x\|^2 + \langle x, y \rangle + \|y\|^2 \end{aligned}$$

1.1 HH-P Orthogonality

Definition. [3] A vector x is said to be orthogonal to y in the sense of Pythagorean if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 .$$

Let $(X, \|.\|)$ be a normed space. Then $x \perp_{HH-P} y \iff \int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2)$.

1.1.1 Properties of HH-P orthogonality

1. HH-P orthogonality satisfies non-degeneracy, simplification, continuity and symmetry.
2. HH-P orthogonality is existent.
3. HH-P orthogonality is unique.
4. HH-P orthogonality is homogeneous if and only if the space is inner-product space.
5. HH-P orthogonality is additive if the space is an inner-product space.

Definition. [4, 5] A vector x is said to be orthogonal to y in the sense of isosceles orthogonal to y in the sense of Isosceles if $\|x + y\| = \|x - y\|$.

1.2 HH-I orthogonality

Let $x, y \in X$ such that $\|(1-t)x + ty\| = \|(1-t)x - ty\|$ a.e. on $[0, 1]$. Then x is said to be HH-I orthogonal to y iff

$$\int_0^1 \|(1-t)x + ty\| \, dt = \int_0^1 \|(1-t)x - ty\| \, dt.$$

1.2.1 Properties of HH-I Orthogonality

1. The HH-I orthogonality satisfies non-degeneracy, simplification, continuity and symmetry properties.
2. HH-I orthogonality is existent.
3. If HH-I orthogonality is homogeneous in a normed space X, then X is an inner-product space.
4. If HH-I orthogonality is additive, then the space is an inner-product space.
5. HH-I orthogonality is neither right additive nor homogeneous.

Definition. [2]

1.3 HH-C Orthogonality

[2] Let $(X, \|.\|)$ be a normed space and $t \in [0,1]$. then $x \in X$ is said to be HH-C orthogonal to to $y \in X$ if and only if

$$\sum_{j=1}^{m} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

satisfying the conditions

$$\sum_{j=1}^{m} \alpha_j \beta_j \gamma_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j \beta J_{\mathrm{J}}^2 = \sum_{j=1}^{m} \alpha_j \gamma_j^2 = 0.$$

HH-P orthogonality is a particular case of HH-C orthogonality

Let us take

$$\sum_{j=1}^{3} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

$$\Rightarrow \alpha_1 \int_0^1 \|(1-t)\beta_1 x + t\gamma_1 y\|^2 \,dt + \alpha_2 \int_0^2 \|(1-t)\beta_2 x + t\gamma_2 y\|^2 \,dt + \alpha_3 \int_0^1 \|(1-t)\beta_3 x + t\gamma_3 y\|^2 \,dt = 0$$

Taking $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$ and $\gamma_2 = 0$, we get

$$-\int_0^1 \|(1-t)x + ty\|^2 \,dt + \int_0^1 \|(1-t)x\|^2 \,dt + \int_0^1 \|ty\|^2 \,dt = 0$$

$$\Rightarrow -\int_0^1 \|(1-t)x + ty\|^2 \,dt + \frac{1}{3}(\|x\|^2 + \|y\|^2 = 0$$

$$\therefore \quad \int_0^1 \|(1-t)x + ty\|^2 \,dt = \frac{1}{3}(\|x\|^2 + \|y\|^2$$

Now

$$\sum_{k=1}^{3} \alpha_j \beta_j \gamma_j = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 = -1, \quad \sum_{j=1}^{m} \alpha_j \beta j^2 \quad = \alpha_1 \beta 1^2 + \alpha_2 \beta 2^2 + \alpha_3 \beta 3^2 = 0$$

$$\text{and} \quad \sum_{j=1}^{m} \alpha_j \gamma_j^2 = \alpha_1\gamma_1^2 + \alpha_2\gamma_2^2 + \alpha_3\gamma_3^2 = 0$$

Which shows that HH-P orthogonality is a particular case of HH-C orthogonality.

HH-I orthogonality is a particular case of HH-C orthogonality

Let us take

$$\sum_{j=1}^{2} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

$$\Rightarrow \alpha_1 \int_0^1 \|(1-t)\beta_1 x + t\gamma_1 y\|^2 \,dt + \alpha_2 \int_0 \|(1-t)\beta_2 x + t\gamma_2 y\|^2 \,dt = 0$$

Taking $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{-1}{2}, \beta_1 = \beta_2 = 1, \gamma_1 = 1, \gamma_2 = -1$, we get

$$\frac{1}{2}\int_0^1 \|(1-t)x + ty\|^2\,dt - \frac{1}{2}\int_0^1 \|(1-t)x - ty\|^2\,dt = 0$$

$$\Rightarrow \int_0^1 \|(1-t)x + ty\|^2\,dt = \int_0^1 \|(1-t)x - ty\|^2\,dt$$

Now

$$\sum_{k=1}^{2} \alpha_j\beta_j\gamma_j = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 = 1, \quad \sum_{k=1}^{2} \alpha_j\beta_j^2 = \alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$$

$$\text{and} \quad \sum_{k=1}^{2} \alpha_j\gamma_j^2 = \alpha_1\gamma_1^2 + \alpha_2\gamma_2^2 = 0$$

1.3.1 Properties of HH-C orthogonality

1. HH-C orthogonality satisfies non-degeneracy, simplification , and continuity property.
2. HH-C orthogonality is not symmetric.
3. HH-C orthogonality is neither additive nor homogeneous.

2 Main Results

Robert Orthogonality in 2-HH Norm

Definition. $x\perp_R y$ if $\forall\lambda \in \mathbb{R}, \quad \|x + \lambda y\| = \|x - \lambda y\|$.

Using the concept of $2 - HH$ norm, $\|(1-t)x + t\lambda y\| = \|(1-t)x - t\lambda y\|$ a.e. on $[0, 1]$, we have the definition of orthogonality in $2 - HH$ norm is as follows:

$$\int_0^1 \|(1-t)x + t\lambda y\|^2\,dt = \int_0^1 \|(1-t)x - t\lambda y\|^2\,dt.$$

Non-degeneracy

If $x\perp_{HH-R}x$. Then

$$\begin{aligned}
\|x\|^2 &= \int_0^1 \|(1-t)x + t\lambda y\|^2\,dt \\
&= \int_0^1 \|(1-t)x - t\lambda y\|^2\,dt \\
&= \int_0^1 \langle(1-t)x + t\lambda y, (1-t)x - t\lambda y\rangle dt \\
&= \|x\|^2 \int_0^1 91-t)^2 dt + \|\lambda x\|^2 \int_0^1 t^2 dt \\
&= \frac{1}{3}\|x\|^2 (1+\lambda^2)
\end{aligned}$$

It is clear that $\|x\| = 0 \Rightarrow x = 0$, which gives the non-degeneracy property.

Simplification

If $x \perp_{HH-R} y$ for any $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}\int_0^1 \|(1-t)\mu x + t\lambda\mu y\|^2 \, dt &= |\mu|^2 \int_0^1 \|(1-t)x + t\lambda y\|^2 \, dt \\ &= |\mu|^2 \int_0^1 \|(1-t)x - t\lambda y\|^2 \, dt \\ &= \int_0^1 \|(1-t)\mu x - t\lambda\mu y\|^2 \, dt\end{aligned}$$

Which shows that $\mu x \perp_{HH-R} \mu y$ for any $\mu \in \mathbb{R}$.

Symmetry

To check the symmetry of $HH - R$ orthogonality,

$$\begin{aligned}\int_0^1 \|(1-t)y + \lambda t x\|^2 &= \frac{1}{3}(\|y\|^2 + \lambda^2 \|x\|^2), \quad \text{but} \\ \int_0^1 \|(1-t)x + \lambda t y\|^2 &= \frac{1}{3}(\|x\|^2 + \lambda^2 \|y\|^2)\end{aligned}$$

$\therefore \quad \int_0^1 \|(1-t)y + \lambda t x\|^2 \neq \int_0^1 \|(1-t)x + \lambda t y\|^2$, which shows that $HH - R$ orthogonality is not symmetric in $2 - HH$ norm.

Continuity

If $x_n \to x, y_n \to y$, and $\forall n \in \mathbb{N} \quad x_n \perp_{HH-R} y_n$. Then by the continuity of norm,

$$\begin{aligned}\int_0^1 \|(1-t)x + \lambda t y\|^2 \, dt &= \int_0^1 \lim_{n\to\infty} \|(1-t)x_n + \lambda t y_n\|^2 \, dt \\ &= \lim_{n\to\infty} \int_0^1 \|(1-t)x_n + \lambda t y_n\|^2 \, dt \\ &= \lim_{n\to\infty} \int_0^1 \|(1-t)x_n - \lambda t y_n\|^2 \, dt \\ &= \int_0^1 \|(1-t)x - t\lambda y\|^2 \, dt\end{aligned}$$

Homogenity

Let x, y be elements of normed space X, and $\lambda, \mu \in \mathbb{R}$

$$\begin{aligned}\int_0^1 \|(1-t)\lambda x + t\mu y\|^2 \, dt &= \int_0^1 \langle (1-t)\lambda x + t\mu y, (1-t)\lambda x + t\mu y\rangle dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt + 2\lambda\mu\langle x, y\rangle \int_0^1 t(1-t)dt + \|\mu y\|^2 \int_0^1 t^2 dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt + \|\mu y\|^2 \int_0^1 t^2 dt \qquad (\because x \perp y) \\ &= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2)\end{aligned}$$

Again,

$$\begin{aligned}\int_0^1 \|(1-t)\lambda x - t\mu y\|^2 \, dt &= \int_0^1 \langle (1-t)\lambda x - t\mu y, (1-t)\lambda x - t\mu y\rangle dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt - 2\lambda\mu\langle x, y\rangle \int_0^1 t(1-t)dt + \|\mu y\|^2 \int_0^1 t^2 dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt + \|\mu y\|^2 \int_0^1 t^2 dt \qquad (\because x \perp y) \\ &= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2)\end{aligned}$$

Which shows that Robert Orthogonality is homogeneous in 2-HH norm if the space is an inner-product space.

Lemma 2.1. *[8](R. C James, Vol.12, p-296) Let $(\mathbb{X}, \|.\|)$ be a normed linear space and $x, y \in \mathbb{X}$*

$$\lim_{\mu\to\infty} \|\mu + k)x + y\| - \|\mu x + y\| = k\,\|x\|\,.$$

Theorem 2.2. *Let X be a normed linear space. Then $\forall x \in X, \quad \exists \mu \in \mathbb{R} : \mu x + y \perp_{HH-R} x$,*

Proof. Let $x, y \in X$ such that $x \neq 0$(for the case of x=0, the proof is trivial). Let us define a function $g : \mathbb{R} \times (0,1) \to \mathbb{R}$ by

$$\begin{aligned}g(\mu, t) &= \|(1-t)(\mu x + y) + \lambda t x\| - \|(1-t)(\mu x + y) - \lambda t x\|\,, \text{ where } \lambda \in \mathbb{R}^+, \mu \in \mathbb{R} \\ &= \|[(1-t)\mu + \lambda t]x + (1-t)y\| - \|[(1-t)\mu - \lambda t]x + (1-t)y\|\end{aligned}$$

and a function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \int_0^1 g(\mu, t)dt.$$

Now

$$\begin{aligned}\lim_{\mu\to\infty} g(\mu, t) &= \lim_{\mu\to\infty} [\|[(1-t)\mu + \lambda t]x + (1-t)y\| - \|[(1-t)\mu - \lambda t]x + (1-t)y\|] \\ &= (1-t) \lim_{\mu\to\infty} \left[\left\|(\mu + \frac{\lambda t}{1-t})x + y\right\| - \left\|(\mu - \frac{\lambda t}{1-t})x + y\right\|\right]\end{aligned}$$

Let $\mu - \frac{\lambda t}{1-t} = \xi$ so that as $\mu \to \infty$, $\xi \to \infty$. Then $\mu + \frac{\lambda t}{1-t} = \xi + \frac{2\lambda t}{1-t}$

$$\begin{aligned}\therefore \quad \lim_{\mu\to\infty} g(\mu,t) &= \lim_{\xi\to\infty}[\left\|(\xi + \frac{2\lambda t)}{1-t})x + y\right\| - \|\xi x + y\|] \\ &= (1-t)\frac{2\lambda t}{1-t} \qquad (\text{ by using Lemma 1.1}) \\ &= 2\lambda t \|x\|\end{aligned}$$

Hence $\lim_{\mu\to\infty} G(\mu) = \lim_{\mu\to\infty}\int_0^1 g(\mu,t)dt = \int_0^1 \lim_{\mu\to\infty} g(\mu,t)dt$ and by continuity of g,

$$\begin{aligned}\lim_{\mu\to\infty} G(\mu) &= \int_0^1 2\lambda t \|x\| \, dt \\ &= \|\lambda x\| > 0\end{aligned}$$

Also for any $t \in (0,1)$,

$$\begin{aligned}\lim_{\mu\to\infty} g(-\mu,t) &= \lim_{\mu\to\infty}[\|[(1-t)(-\mu)+\lambda t]x + (1-t)y\| - \|[(1-t)(-\mu)-\lambda t]x + (1-t)y\|] \\ &= \lim_{\mu\to\infty}[\|[(1-t)\mu-\lambda t]x - (1-t)y\| - \|[(1-t)\mu+\lambda t]x - (1-t)y\|] \\ &= (1-t)\lim_{\mu\to\infty}\left[\left\|(\mu - \frac{\lambda t}{1-t})x - y\right\| - \left\|(\mu + \frac{\lambda t}{1-t})x - y\right\|\right]\end{aligned}$$

Suppose $\mu + \frac{\lambda t}{1-t} = \xi$ so that as $\mu \to \infty$. $\xi \to \infty$ and $\mu - \frac{2\lambda t}{1-t} = \xi - \frac{2\lambda t}{1-t}$

$$\begin{aligned}\therefore \quad \lim_{\mu\to\infty} g(-\mu,t) &= (1-t)\lim_{\xi\to\infty}\left[\left\|(\xi - \frac{2\lambda t}{1-t})x - y\right\| - \|\xi x - y\|\right] \\ &= (1-t)\frac{(-2\lambda t)}{1-t}\|x\| \qquad (\text{ by using Lemma 1.1}) \\ &= -2\lambda t \|x\|\end{aligned}$$

By the continuity of g, we have

$$\lim_{\mu\to\infty} G(-\mu) = \lim_{\mu\to\infty}\int_0^1 g(-\mu,t)dt = \int_0^1 \lim_{\mu\to\infty} g(-\mu,t)dt = \int_0^1 -2\lambda \|x\| \, dt = -\lambda \|x\| < 0.$$

Sine G is continuous, so $\exists \mu_0 \in \mathbb{R} : G(\mu_0) = 0$.

$$\text{Hence} \int_0^1 \|(1-t)(\mu_0 x + y) + \lambda t x\|^2 \, dt = \int_0^1 \|(1-t)(\mu_0 x + y) - \lambda t x\|^2 \, dt.$$

□

Data Availability

There is not use of any data for the completion of this study.

Conflict of Interest

We authors do no have a conflict of interest for the publication of article.

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