Robert Orthogonality in Normed Linear Spaces
Via 2-HH Norm

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Abstract

The p-HH norms on $X^2$ were introduced by Kikianty and Dragomir in 2008. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Robert orthogonality in normed spaces and discuss some properties of this orthogonality.

Keywords: Robert orthogonality, p-HH norm, Isosceles orthogonality, Pythagorean orthogonality, Hermite-Hadamards inequality.

1 Introduction

The p-HH norms are equivalent to p-norms on $X^2$, as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:

**Definition.** [6, 9] For any convex function $f : [a, b] \rightarrow \mathbb{R} ([a, b] \subset \mathbb{R}$, the Hermite-Hadamard’s inequality is defined as

$$(b - a)f \left( \frac{a + b}{2} \right) \leq \int_a^b f(t)dt \leq (b - a) \left[ \frac{f(a) + f(b)}{2} \right].$$

This inequality has been extended (see-12) for convex function $f : [x, y] \rightarrow \mathbb{R}$, where $[x, y] = \{(1 - t)x + ty, t \in [0, 1]\}$. In that case Hermite-Hadamards integral inequality becomes

$$f \left( \frac{x + y}{2} \right) \leq \int_0^1 f [(1 - t)x + ty] dt \leq \frac{f(x) + f(y)}{2} \quad \text{......(1).}$$

Using the convexity of $f(x) = \|x\|^p$ $(x \in X, p \geq 1)$ and relation (1) we have

$$\left\| \frac{x + y}{2} \right\| \leq \left[ \int_0^1 \| (1 - t)x + ty \|^p dt \right]^{\frac{1}{p}} \leq \frac{1}{2^p} (\|x\|^p + \|y\|^p)^{\frac{1}{p}}.$$

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Definition. (Generalized Geometric Mean)[9] The generalized geometric mean of order \( p \) of \( x \) and \( y \) \((x, y > 0, p \text{ is an extended real number})\) is defined by

\[
L[p](x, y) = \begin{cases} 
\frac{1}{p+1} \left( \frac{x^{p+1} - y^{p+1}}{y - x} \right)^{\frac{1}{p}}, & \text{if } p \neq -1, 0, \pm \infty \\
\frac{y - x}{\log y - \log x}, & \text{if } p = -1 \\
\frac{1}{e} \left( \frac{y^p}{x^p} \right)^{\frac{1}{p}}, & \text{if } p = 0 \\
\max\{x, y\}, & \text{if } p = \infty \\
\min\{x, y\}, & \text{if } p = -\infty
\end{cases}
\]

and \( L[p](x, x) = x \)

A function \( d : X \times X \to \mathbb{R} \) is called a metric on \( X \) if it satisfies the following conditions:

1. \( \forall x, y \in X, \ d(x, y) \geq 0. \)
2. \( \forall x, y \in X, \ d(x, y) = 0 \iff x = y. \)
3. \( \forall x, y \in X, \ d(x, y) = d(y, x). \)
4. \( \forall x, y, z \in X, \ d(x, y) \leq d(x, z) + d(z, y) \)

[1] A vector space \( X \) is said to be normed space if there is a mapping \( \| \cdot \| : X \to \mathbb{R} \) on \( X \) satisfying the properties:

1. \( \| x \| \geq 0, \ \| x \| = 0 \iff x = 0. \)
2. \( \| \lambda x \| = |\lambda| \| x \|. \)
3. \( \| x + y \| \leq \| x \| + \| y \|. \)

An inner-product space on a vector space \( X \) is a mapping \( \langle \cdot, \cdot \rangle : X \times X \to \mathbb{K} \), where \( \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C} \) satisfying the following properties:

1. \( \forall x, y, z \in X, \ \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle. \)
2. \( \forall x, y \in X \text{ and } \alpha \in \mathbb{K}, \ \langle \alpha x, \rangle = \alpha \langle x, y \rangle. \)
3. \( \forall x, y \in X, \ \langle x, y \rangle = \langle y, x \rangle. \)
4. \( \forall x \in X, \ \langle x, x \rangle \geq 0, \langle x, x \rangle = 0 \iff x = 0. \)

An inner-product on \( X \) defines a norm on \( X \) by \( \| x \|^2 = \langle x, x \rangle \). Every innerproduct spaces are normed spaces, but the converse may not be true. A best example of normed space which is not an inner-product space is \( l^p = \{ (x_n), x_n \in \mathbb{R} : \sum |x_n| < \infty \} \) for \( p \neq 2 \).

Definition. [7] The \( p-HH \) norm on \( X^2 = X \times X \) is defined by

\[
\|(x, y)\|_{p-HH} = \left( \int_0^1 \|(1 - t)x + ty\|^p \, dt \right)^{\frac{1}{p}}
\]

for any \( x, y \in X^2 \) and \( 1 \leq p < \infty \).

The 2-HH norm is defined as follows:

\[
\|(x, y)\|_{2-HH}^2 = \int_0^1 \|(1 - t)x + ty\|^2 \, dt
= \frac{1}{3} \|x\|^2 + \langle x, y \rangle + \|y\|^2
\]
1.1 HH-P Orthogonality

Definition. [3] A vector \(x\) is said to be orthogonal to \(y\) in the sense of Pythagorean if
\[
\|x - y\|^2 = \|x\|^2 + \|y\|^2.
\]
Let \((X, \|\|)\) be a normed space. Then \(x \perp_{\text{HH-P}} y \iff \int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2)\).

1.1.1 Properties of HH-P orthogonality

1. HH-P orthogonality satisfies non-degeneracy, simplification, continuity and symmetry.
2. HH-P orthogonality is existent.
3. HH-P orthogonality is unique.
4. HH-P orthogonality is homogeneous if and only if the space is inner-product space.
5. HH-P orthogonality is additive if the space is an inner-product space.

Definition. [4, 5] A vector \(x\) is said to be orthogonal to \(y\) in the sense of isosceles orthogonal to \(y\) in the sense of Isosceles if \(\|x + y\| = \|x - y\|\).

1.2 HH-I orthogonality

Let \(x, y \in X\) such that \(\|(1-t)x + ty\| = \|(1-t)x - ty\|\) a.e. on \([0, 1]\). Then \(x\) is said to be HH-I orthogonal to \(y\) iff
\[
\int_0^1 \|(1-t)x + ty\| \, dt = \int_0^1 \|(1-t)x - ty\| \, dt.
\]

1.2.1 Properties of HH-I Orthogonality

1. The HH-I orthogonality satisfies non-degeneracy, simplification, continuity and symmetry properties.
2. HH-I orthogonality is existent.
3. If HH-I orthogonality is homogeneous in a normed space \(X\), then \(X\) is an inner-product space.
4. If HH-I orthogonality is additive, then the space is an inner-product space.
5. HH-I orthogonality is neither right additive nor homogeneous.

Definition. [2]
1.3 HH-C Orthogonality

[2] Let \((X, \| \cdot \|)\) be a normed space and \(t \in [0, 1] \). then \(x \in X\) is said to be HH-C orthogonal to \(y \in X\) if and only if

\[
\sum_{j=1}^{m} \alpha_j \int_0^1 \| (1 - t) \beta_j x + t \gamma_j y \|^2 = 0
\]

satisfying the conditions

\[
\sum_{j=1}^{m} \alpha_j \beta_j \gamma_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j \beta_j \gamma_j^2 = 0.
\]

HH-P orthogonality is a particular case of HH-C orthogonality

Let us take

\[
\sum_{j=1}^{m} \alpha_j \int_0^1 \| (1 - t) \beta_j x + t \gamma_j y \|^2 = 0
\]

\[
\Rightarrow \alpha_1 \int_0^1 \| (1 - t) \beta_1 x + t \gamma_1 y \|^2 dt + \alpha_2 \int_0^1 \| (1 - t) \beta_2 x + t \gamma_2 y \|^2 dt + \alpha_3 \int_0^1 \| (1 - t) \beta_3 x + t \gamma_3 y \|^2 dt = 0
\]

Taking \(\alpha_1 = -1, \alpha_2 = \alpha_3 = 1, \beta_1 = \beta_2 = 1, \beta_3 = 0, \gamma_1 = \gamma_3 = 1\) and \(\gamma_2 = 0\), we get

\[
- \int_0^1 \| (1 - t) x + t y \|^2 dt + \int_0^1 \| (1 - t) x \|^2 dt + \int_0^1 \| t y \|^2 dt = 0
\]

\[
\Rightarrow - \int_0^1 \| (1 - t) x + t y \|^2 dt + \frac{1}{3} (\|x\|^2 + \|y\|^2) = 0
\]

\[
\therefore \int_0^1 \| (1 - t) x + t y \|^2 dt = \frac{1}{3} (\|x\|^2 + \|y\|^2)
\]

Now

\[
\sum_{k=1}^{3} \alpha_j \beta_j \gamma_j = \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 + \alpha_3 \beta_3 \gamma_3 = -1, \quad \sum_{j=1}^{m} \alpha_j \beta_j \gamma_j^2 = \alpha_1 \beta_1 \gamma_1^2 + \alpha_2 \beta_2 \gamma_2^2 + \alpha_3 \beta_3 \gamma_3^2 = 0
\]

and \(\sum_{j=1}^{m} \alpha_j \gamma_j^2 = \alpha_1 \gamma_1^2 + \alpha_2 \gamma_2^2 + \alpha_3 \gamma_3^2 = 0\)

Which shows that HH-P orthogonality is a particular case of HH-C orthogonality.

HH-I orthogonality is a particular case of HH-C orthogonality

Let us take

\[
\sum_{j=1}^{m} \alpha_j \int_0^1 \| (1 - t) \beta_j x + t \gamma_j y \|^2 = 0
\]

\[
\Rightarrow \alpha_1 \int_0^1 \| (1 - t) \beta_1 x + t \gamma_1 y \|^2 dt + \alpha_2 \int_0^1 \| (1 - t) \beta_2 x + t \gamma_2 y \|^2 dt = 0
\]
Taking \( \alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{2}, \beta_1 = 1, \gamma_1 = 1, \gamma_2 = -1 \), we get
\[
\frac{1}{2} \int_0^1 \| (1 - t)x + ty \|^2 dt - \frac{1}{2} \int_0^1 \| (1 - t)x - ty \|^2 dt = 0
\]
\[\Rightarrow \int_0^1 \| (1 - t)x + ty \|^2 dt = \int_0^1 \| (1 - t)x - ty \|^2 dt \]

Now
\[
\sum_{k=1}^2 \alpha_j \beta_j \gamma_j = \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 = 1, \quad \sum_{k=1}^2 \alpha_j \beta_j^2 = \alpha_1 \beta_1^2 + \alpha_2 \beta_2^2 = 0
\]
and
\[
\sum_{k=1}^2 \alpha_j \gamma_j^2 = \alpha_1 \gamma_1^2 + \alpha_2 \gamma_2^2 = 0
\]

1.3.1 Properties of HH-C orthogonality

1. HH-C orthogonality satisfies non-degeneracy, simplification, and continuity property.
2. HH-C orthogonality is not symmetric.
3. HH-C orthogonality is neither additive nor homogeneous.

2 Main Results

Robert Orthogonality in 2-HH Norm

Definition. \( x \perp_R y \) if \( \forall \lambda \in \mathbb{R}, \quad \| x + \lambda y \| = \| x - \lambda y \| \).

Using the concept of 2−HH norm, \( \| (1 - t)x + t\lambda y \| = \| (1 - t)x - t\lambda y \| \) a.e. on \([0, 1]\), we have the definition of orthogonality in 2−HH norm is as follows:
\[
\int_0^1 \| (1 - t)x + t\lambda y \|^2 dt = \int_0^1 \| (1 - t)x - t\lambda y \|^2 dt.
\]

Non-degeneracy

If \( x \perp_{HH-R} x \). Then
\[
\| x \|^2 = \int_0^1 \| (1 - t)x + t\lambda y \|^2 dt
\]
\[
= \int_0^1 \| (1 - t)x - t\lambda y \|^2 dt
\]
\[
= \int_0^1 \langle (1 - t)x + t\lambda y, (1 - t)x - t\lambda y \rangle dt
\]
\[
= \| x \|^2 \int_0^1 (1 - t)^2 dt + \| \lambda x \|^2 \int_0^1 t^2 dt
\]
\[
= \frac{1}{3} \| x \|^2 (1 + \lambda^2)
\]
It is clear that \( \|x\| = 0 \Rightarrow x = 0 \), which gives the non-degeneracy property.

**Simplification**

If \( x \perp_{HH-R} y \) for any \( \lambda, \mu \in \mathbb{R} \),

\[
\int_0^1 \|(1-t)\mu x + t\lambda \mu y\|^2 dt = |\mu|^2 \int_0^1 \|(1-t)x + t\lambda y\|^2 dt
\]

\[
= |\mu|^2 \int_0^1 \|(1-t)x - t\lambda y\|^2 dt
\]

\[
= \int_0^1 \|(1-t)\mu x - t\lambda \mu y\|^2 dt
\]

Which shows that \( \mu x \perp_{HH-R} \mu y \) for any \( \mu \in \mathbb{R} \).

**Symmetry**

To check the symmetry of \( HH - R \) orthogonality,

\[
\int_0^1 \|(1-t)y + \lambda tx\|^2 = \frac{1}{3}(\|y\|^2 + \lambda^2 \|x\|^2), \quad \text{but}
\]

\[
\int_0^1 \|(1-t)x + \lambda ty\|^2 = \frac{1}{3}(\|x\|^2 + \lambda^2 \|y\|^2)
\]

\[
\therefore \int_0^1 \|(1-t)y + \lambda tx\|^2 \neq \int_0^1 \|(1-t)x + \lambda ty\|^2, \quad \text{which shows that } HH - R \text{ orthogonality is not symmetric in } 2 - HH \text{ norm.}
\]

**Continuity**

If \( x_n \to x, y_n \to y, \) and \( \forall n \in \mathbb{N} \; x_n \perp_{HH-R} y_n \). Then by the continuity of norm,

\[
\int_0^1 \|(1-t)x + \lambda ty\|^2 dt = \int_0^1 \lim_{n \to \infty} \|(1-t)x_n + \lambda ty_n\|^2 dt
\]

\[
= \lim_{n \to \infty} \int_0^1 \|(1-t)x_n + \lambda ty_n\|^2 dt
\]

\[
= \lim_{n \to \infty} \int_0^1 \|(1-t)x_n - \lambda ty_n\|^2 dt
\]

\[
= \int_0^1 \|(1-t)x - t\lambda y\|^2 dt
\]
Homogenity

Let \( x, y \) be elements of normed space \( X \), and \( \lambda, \mu \in \mathbb{R} \)

\[
\int_0^1 \|(1-t)\lambda x + t\mu y\|^2 \, dt = \int_0^1 \langle (1-t)\lambda x + t\mu y, (1-t)\lambda x + t\mu y \rangle \, dt
\]

\[
= \|\lambda x\|^2 \int_0^1 (1-t)^2 \, dt + 2\lambda \mu \langle x, y \rangle \int_0^1 t(1-t) \, dt + \|\mu y\|^2 \int_0^1 t^2 \, dt
\]

\[
= \|\lambda x\|^2 \int_0^1 (1-t)^2 \, dt + \|\mu y\|^2 \int_0^1 t^2 \, dt \quad (\because x \bot y)
\]

\[
= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2)
\]

Again,

\[
\int_0^1 \|(1-t)\lambda x - t\mu y\|^2 \, dt = \int_0^1 \langle (1-t)\lambda x - t\mu y, (1-t)\lambda x - t\mu y \rangle \, dt
\]

\[
= \|\lambda x\|^2 \int_0^1 (1-t)^2 \, dt - 2\lambda \mu \langle x, y \rangle \int_0^1 t(1-t) \, dt + \|\mu y\|^2 \int_0^1 t^2 \, dt
\]

\[
= \|\lambda x\|^2 \int_0^1 (1-t)^2 \, dt + \|\mu y\|^2 \int_0^1 t^2 \, dt \quad (\because x \bot y)
\]

\[
= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2)
\]

Which shows that Robert Orthogonality is homogeneous in 2-HH norm if the space is an inner-product space.

**Lemma 2.1.** \([8](R. C James, Vol. 12, p-296)\) Let \((X, \|\|.\|)\) be a normed linear space and \( x, y \in X \)

\[
\lim_{\mu \to \infty} \|\mu + k\rangle x + y \| - \|\mu x + y\| = k \|x\|.
\]

**Theorem 2.2.** Let \( X \) be a normed linear space. Then \( \forall x \in X, \exists \mu \in \mathbb{R} : \mu x + y \bot_{HH-R} x \)

*Proof.* Let \( x, y \in X \) such that \( x \neq 0 \) (for the case of \( x=0 \), the proof is trivial). Let us define a function \( g : \mathbb{R} \times (0, 1) \to \mathbb{R} \) by

\[
g(\mu, t) = \|(1-t)(\mu x + y) + \lambda t x\| - \|(1-t)(\mu x + y) - \lambda t x\|, \quad \text{where} \quad \lambda \in \mathbb{R}^+, \mu \in \mathbb{R}
\]

\[
= \|[1-t] \mu + \lambda t\rangle x + (1-t) y\| - \|[1-t] \mu - \lambda t\rangle x + (1-t) y\|
\]

and a function \( G : \mathbb{R} \to \mathbb{R} \) by

\[
G(x) = \int_0^1 g(\mu, t) \, dt.
\]

Now

\[
\lim_{\mu \to \infty} g(\mu, t) = \lim_{\mu \to \infty} \|[1-t] \mu + \lambda t\rangle x + (1-t) y\| - \|[1-t] \mu - \lambda t\rangle x + (1-t) y\|
\]

\[
= (1-t) \lim_{\mu \to \infty} \left\| \left(\mu + \frac{\lambda t}{1-t}\right)x + y \right\| - \left\| \left(\mu - \frac{\lambda t}{1-t}\right)x + y \right\|
\]
Let $\mu - \frac{\lambda t}{1-t} = \xi$ so that as $\mu \to \infty$, $\xi \to \infty$. Then $\mu + \frac{\lambda t}{1-t} = \xi + \frac{2\lambda t}{1-t}$

\[ \therefore \lim_{\mu \to \infty} g(\mu, t) = \lim_{\xi \to \infty} \left\| \left( \xi + \frac{2\lambda t}{1-t} \right)x + y \right\| - \left\| \xi x + y \right\| \]

\[ = (1-t)\frac{2\lambda t}{1-t} \quad \text{(by using Lemma 1.1)} \]

\[ = 2\lambda t \left\| x \right\| \]

Hence $\lim_{\mu \to \infty} G(\mu) = \lim_{\mu \to \infty} \int_0^1 g(\mu, t)dt = \int_0^1 \lim_{\mu \to \infty} g(\mu, t)dt$ and by continuity of $g$,

\[ \lim_{\mu \to \infty} G(\mu) = \int_0^1 2\lambda t \left\| x \right\| dt \]

\[ = \left\| \lambda x \right\| > 0 \]

Also for any $t \in (0, 1)$,

\[ \lim_{\mu \to \infty} g(-\mu, t) = \lim_{\mu \to \infty} \left[ \left\| \left( 1-t \right)(-\mu) + \lambda t \right\| x + (1-t)y \right\| - \left\| \left( 1-t \right)(-\mu) - \lambda t \right\| x + (1-t)y \right\| \]

\[ = \lim_{\mu \to \infty} \left[ \left\| \left( 1-t \right)\mu - \lambda t \right\| x - (1-t)y \right\| - \left\| \left( 1-t \right)\mu + \lambda t \right\| x - (1-t)y \right\| \]

\[ = (1-t)\lim_{\mu \to \infty} \left[ \left\| \left( \mu - \frac{\lambda t}{1-t} \right)x - y \right\| - \left\| \left( \mu + \frac{\lambda t}{1-t} \right)x - y \right\| \right] \]

Suppose $\mu + \frac{\lambda t}{1-t} = \xi$ so that as $\mu \to \infty$, $\xi \to \infty$ and $\mu - \frac{2\lambda t}{1-t} = \xi - \frac{2\lambda t}{1-t}$

\[ \therefore \lim_{\mu \to \infty} g(-\mu, t) = (1-t)\lim_{\xi \to \infty} \left[ \left\| \left( \xi - \frac{2\lambda t}{1-t} \right)x - y \right\| - \left\| \xi x - y \right\| \right] \]

\[ = (1-t)\left( -\frac{2\lambda t}{1-t} \right) \left\| x \right\| \quad \text{(by using Lemma 1.1)} \]

\[ = -2\lambda t \left\| x \right\| \]

By the continuity of $g$, we have

\[ \lim_{\mu \to \infty} G(-\mu) = \lim_{\mu \to \infty} \int_0^1 g(-\mu, t)dt = \int_0^1 \lim_{\mu \to \infty} g(-\mu, t)dt = \int_0^1 -2\lambda \left\| x \right\| dt = -\lambda \left\| x \right\| < 0. \]

Sine $G$ is continuous, so $\exists \mu_0 \in \mathbb{R} : G(\mu_0) = 0.$

\[ \text{Hence} \int_0^1 \left\| \left( 1-t \right)(\mu_0 x + y) + \lambda t x \right\|^2 dt = \int_0^1 \left\| \left( 1-t \right)(\mu_0 x + y) - \lambda t x \right\|^2 dt. \]

\[ \square \]

**Data Availability**

There is not use of any data for the completion of this study.
Conflict of Interest

We authors do no have a conflict of interest for the publication of article.

Acknowledgment

I am thankful to my PhD supervisor Prof. Dr. Prakash Muni Bajryacharya, Prof. Dr. Kedar Nath Uprety, the head of the Central Department of Mathematics and my respected professors for their continuous support and feedback during my study.

References


