ON THE EIGEN BUCKLING MODES AND STATIC BUCKLING OF CLAMPED CIRCULAR CYLINDRICAL SHELLS

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ABSTRACT:
Elastic instability of structural materials of which buckling is a critical example has been in the search light of investigations for a long time now. In this paper, we embark on a similar investigation involving deterministically imperfect but clamped finite circular cylindrical shells. As the governing equations are strictly nonlinear, we employ asymptotic and perturbation procedures in a purely analytical approach to solve the problem. The imperfection and buckling modes are expressed in a form of Fourier series and also in a form compatible with the boundary conditions.
The results show, among other things, that the number of buckling modes increases with higher order perturbations. Using only the buckling modes in the shape of imperfection, the static buckling load was also derived and was seen to be in a form characteristic of all cubic structures of which the cylindrical shells are practical examples.

Keywords and Phrases: Buckling modes, static buckling, clamped, circular cylindrical shells, perturbation and asymptotic methods.
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1. INTRODUCTION

This analysis is concerned with analytical investigation of the emergence of several buckling modes as well as static buckling load of imperfect finite circular cylindrical shells with clamped boundary conditions. Circular cylindrical shells under static and dynamic loading conditions have been investigated for a long time now. In one of such early investigations, Batdorf [1], presented a simplified method of elastic –stability analysis for cylindrical shells while Amazigo and Frazer [2] studied the buckling, under external pressure, of cylindrical shells with dimple-shaped initial imperfections. In yet another study, Budiansky and Amazigo [3] investigated the initial post buckling behaviour of cylindrical shells under external pressure while Lockhart and Amazigo [4] similarly investigated the dynamic buckling of externally pressurized imperfect cylindrical shells. Relatively-recent studies on the subject matter include Hu and Burgueño [5] who investigated elastic post-buckling response of axially-loaded cylindrical shells with seeded geometric imperfection design, while Kriegesmann et al. [6] studied size dependent probabilistic design of axially compressed cylindrical shells. Investigations by Castro et al. [7] and Burgueño [8] on the subject matter were particularly insightful.

In this study, we shall assume cylindrical shells with arbitrary stress-free initial displacement (which serves as the initial imperfection), while the cylindrical shells, as a whole, are subjected to lateral or hydrostatic pressure. The magnitude of the imperfection is assumed small compared to shell thickness, and, as in Ette and Chukwuchekwa [9], we shall adopt asymptotic and perturbation techniques where all series expansions are made in terms of the small magnitude of the imperfection ϵ.

In most of the earlier investigations such as the ones by Koiter [10], it was assumed that the imperfections could be taken in the shape of the classical buckling mode. This assertion is not fully adhered to in this study. Rather, a judicious use is made of representations of the imperfection and the normal displacement in terms of Fourier series and in a manner compatible with the boundary conditions. Since one of our objectives is anchored on determining the eigen buckling modes, we are not necessarily restricting the buckling modes to be strictly in the shape of either the imperfection or classical buckling mode. In this way, we are able to account for all the possible eigen buckling modes within the limit of accuracy retained. Other similar investigations are seen in [11]-[18]

2. CIRCULAR CYLINDER EQUATIONS

The associated Karman-Donnell equations of equilibrium and compatibility equation [4] governing the normal deflection $W(X,Y)$ and Airy stress function $F(X,Y)$ for cylindrical shells of length L, radius R, thickness h, bending stiffness $D = \frac{Eh^3}{12(L-\upsilon^2)}$, (where E and ν are the Young's modulus and Poisson ratio respectively), mass per unit area ρ, subjected to external pressure per unit area P are

$$D\nabla^4 W + \frac{1}{R}F,_{XX} = \bar{S}(W + \overline{W}, F) - P \tag{2.1}$$

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{R}W,_{XX} = -S\left(W, \frac{1}{2}W + \overline{W}\right) \tag{2.2a}$$

$$0 < X < L, \quad 0 < Y < R \tag{2.2b}$$

$$W = W,_X = 0 \quad \text{at} \quad X = 0, \pi \tag{2.2c}$$

where X and Y are the axial and circumferential coordinates respectively and $\overline{W}(X,Y)$ is a twice-differentiable stress-free and time independent imperfection. Except and perhaps some terms on the right-hand sides of (2.1) and (2.2a, b, c) a subscript placed after a comma indicates partial differentiation, while $\bar{S}$ is the symmetric bilinear operator given by

$$\bar{S}(P,Q) = P,_{XX}\, Q,_{YY} + P,_{YY}\, Q,_{XX} - 2P,_{XY}\, Q,_{XY} \tag{2.3a}$$

and ∇^4 is the bi-harmonic operator defined by

$$\nabla^4 = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2 \tag{2.3b}$$

3. NONDIMENSIONALIZATION OF THE GOVERNING EQUATIONS

We now introduce the following non dimensional quantities

$$x = \frac{X\pi}{L}, \qquad y = \frac{2\pi}{R}, \qquad \epsilon\overline{w} = \frac{\overline{W}}{h}, \qquad w = \frac{W}{h} \tag{3.1a}$$

$$\lambda = \frac{L^2 RP}{\pi^2 D}, \qquad A = \frac{L^2\sqrt{12(1-\upsilon^2)}}{\pi RL}, \qquad \xi = \frac{L^2}{\pi^2 R^2} \tag{3.1b}$$

$$K(\xi) = \frac{A^2}{(1+\xi)^2}, \qquad H = \frac{h}{R}, \quad 0 < \epsilon \ll 1 \tag{3.1c}$$

We shall assume clamped boundary conditions and shall neglect boundary layer effects by assuming that the pre-buckling deflection is constant so that we let

$$F = -\frac{PR}{2}\left(X^2 + \frac{\alpha Y^2}{2}\right) + \left(\frac{Eh^2 L}{\pi^2 R(1+\xi)^2}\right) f \tag{3.2}$$

$$W = \frac{PR^2\left(1-\frac{\alpha\upsilon}{2}\right)}{Eh} + hw \tag{3.3}$$

where is P the applied static load and λ is the non-dimensional load parameter. The first terms on the right-hand sides of (3.2) and (3.3) are pre-buckling approximations, while the parameter α shall take the value $\alpha = 1$, if pressure contributes to axial stress through the ends, whereas $\alpha = 0$, if pressure only acts laterally. On substituting (3.2) and (3.3) into (2.1) and (2.2) and simplifying, we get

$$\overline{\nabla}^4 w - K(\xi) f_{,xx} + \lambda\left[\frac{\alpha}{2}(w+\epsilon\overline{w})_{,xx} + \xi(w+\epsilon\overline{w})_{,yy}\right] = -HK(\xi)S(w+\epsilon\overline{w},\ f) \tag{3.4}$$

and

$$\overline{\nabla}^4 f - (1+\xi)^2 w_{,xx} = -H(1+\xi)^2 S\left(w, \frac{1}{2}w + \epsilon\overline{w}\right) \tag{3.5a}$$

$$0 < x < \pi, \quad 0 < y < 2\pi, \quad 0 < \epsilon \ll 1 \tag{3.5b}$$

$$w = w_{,X} = 0 \quad \text{at} \quad x = 0, \pi \tag{3.5c}$$

and where,

$$\overline{\nabla}^4 = \left(\frac{\partial^2}{\partial x^2} + \xi\frac{\partial^2}{\partial y^2}\right)^2 \tag{3.5d}$$

$$S(P,Q) = P_{,xx}Q_{,yy} + P_{,yy}Q_{,xx} - 2P_{,xy}Q_{,xy} \tag{3.5e}$$

4. CLASSICAL BUCKLING LOAD

The classical buckling load λ_C is the load required to buckle the associated perfect linear structure and the required equations, from (3.4) and (3.5a) are

$$\overline{\nabla}^4 w - K(\xi) f_{,xx} + \lambda\left[\frac{\alpha}{2}w_{,xx} + \epsilon\overline{w}_{,yy}\right] = 0 \tag{4.1}$$

and

$$\overline{\nabla}^4 f - (1+\xi)^2 w_{,xx} = 0 \tag{4.2a}$$

$$0 < x < \pi, \quad 0 < y < 2\pi, \quad w = w_{,X} = 0 \quad \text{at} \quad x = 0, \pi \tag{4.2b}$$

Based on the boundary conditions, the general solution to (4.1) and (4.2a,b) will be a superposition of the form

$$(w, f) = (\alpha_{rk}, \beta_{rk})(1 - \cos 2rx)\sin ky \tag{4.3}$$

where,

$$(\alpha_{rk}, \beta_{rk}) \neq (0,0)$$

On substituting (4.3) into the left hand sides of (4.1) and (4.2a), we get

$$(\overline{\nabla}^4 w, \overline{\nabla}^4 f) = (\alpha_{rk}, \beta_{rk})[-(16r^4 + 8\xi r^2 k^2)\cos 2rx \sin ky \tag{4.4}$$

Now, substituting for $\overline{\nabla}^4 f$ from (4.4) into (4.2a) and after simplifying by first multiplying the resultant equation by $\cos 2mx \sin ny$ (for m, n fixed and positive integers), and integrating from 0 to π (for x) and from 0 to 2π (for y), we get

$$\beta_{mn} = \frac{4(1+\xi)^2 m^2 \alpha_{mn}}{4(m^2+n^2\xi)^2} \tag{4.5}$$

We next substitute (4.5) into (4.1), using $\overline{\nabla}^4 w$ as in (4.4), multiply by $\cos 2mx \sin ny$ and integrate as before to get

$$\alpha_{mn}[(4m^2 + n^2\xi)^2 - \lambda(2m^2\alpha - n^2\xi)] + 4K(\xi)m^2\beta_{mn} = 0 \tag{4.6}$$

By simplifying, we get

$$\lambda = \frac{(4m^2(1+\xi))^2 K(\xi) + (4m^2+n^2\xi)^4}{(2m^2\alpha+n^2\xi)(4m^2+n^2\xi)^2} \tag{4.7a}$$

That is

$$\lambda = \frac{(4m^2+n^2\xi)^4 - (4mA)^2}{(2m^2\alpha+n^2\xi)(4m^2+n^2\xi)^2} \tag{4.7b}$$

where we have substituted for $K(\xi)$ to get (4.7b).

Usually, m and n take the values $m = 1,2,3, \dots$ and $n = 0,1,2,3, \dots$

Batdorf [1] had assumed that n varies continuously, and so he minimized λ with respect to n. Thus, if $\hat{n}$ is the value of n that minimizes λ, then, the value of λ at $\hat{n}$ was taken as the classical buckling load λ_C. Thus, in this case, we get

$$\lambda_C = \frac{(4m^2+\hat{n}^2\xi)^4 - (4m\Theta)^2}{(2m^2\alpha+\hat{n}^2\xi)(4m^2+\hat{n}^2\xi)^2} \tag{4.8}$$

If $m = 1$ is the nontrivial values of m and we let $\zeta = \hat{n}^2\xi$, then

$$\lambda_C = \frac{(4+\zeta)^4 - (4\Theta)^2}{(2\alpha+\zeta)(4+\zeta)^2} \tag{4.9}$$

Thus, the corresponding deflection and Airy Stress function are

$$(w, f) = \left(1,\ \frac{4(1+\zeta)^2}{(4+\zeta)^2}\right)\alpha_{1\hat{n}}(1 - \cos 2x)\sin \hat{n}y \tag{4.10}$$

5. SOLUTION OF THE FULL STATIC PROBLEM

In [4] where simply-supported boundary conditions were used, it was assumed that if the edge effects could be neglected, then the imperfection $\overline{w}(x, y)$ could be taken in the form of Fourier series expansions thus:

$$\overline{w}(x, y) = \bar{a}_{1n} \sin x \sin ny + \sum_{\substack{m=1,k=0 \\ (m,k)\neq(1,n)}}^{\infty} (\bar{a}_{mk} \sin ky + \bar{b}_{mk} \cos ky) \sin mx \tag{5.1a}$$

or

$$\overline{w}(x, y) = \sum_{m=1,k=0}^{\infty} (\bar{a}_{mk} \sin ky + \bar{b}_{mk} \cos ky) \sin mx \tag{5.1b}$$

with,

$$\bar{b}_{1n} = 0 \tag{5.1c}$$

In our investigation concerning clamped boundary conditions, we shall take

$$\overline{w}(x, y) = \bar{a}(1 - \cos 2mx) \sin ny \tag{5.2}$$

and also assume the following asymptotic expansions

$$(w, f) = \sum_{i=1}^{\infty} \binom{w^{(i)}}{f^{(i)}} \epsilon^i \tag{5.3}$$

On substituting (5.3) into (3.5a) and (3.4) and equating the orders of ϵ (beginning with (3.5a) in each case), we get

$$O(\epsilon): \begin{cases} L^{(1)}\left(f^{(1)}, w^{(1)}\right) \equiv \overline{\nabla}^4 f^{(1)} - (1+\xi)^2 w,_{xx}^{(1)} = 0 & (5.4) \\ \overline{\nabla}^4 w^{(1)} - K(\xi) f,_{xx}^{(1)} + \lambda\left[\frac{\alpha}{2}\left(w^{(1)} + \overline{w}\right),_{xx} + \xi\left(w^{(1)} + \overline{w}\right),_{yy}\right] = 0 & (5.5) \end{cases}$$

$$O(\epsilon^2): \begin{cases} L^{(1)}\left(f^{(2)}, w^{(2)}\right) = -(1+\xi)^2 H\left[S\left(w^{(1)}, \frac{1}{2} w^{(1)}\right) + S\left(w^{(1)}, \overline{w}\right)\right] & (5.6) \\ L^{(2)}\left(f^{(2)}, w^{(2)}\right) \equiv \overline{\nabla}^4 w^{(2)} - K(\xi) f,_{xx}^{(2)} + \lambda\left[\frac{\alpha}{2} w^{(2)},_{xx} + \epsilon w^{(2)},_{yy}\right] & \\ \qquad = -HK(\xi)\left[S\left(w^{(1)}, f^{(1)}\right) + S\left(\overline{w}, f^{(1)}\right)\right] & (5.7) \end{cases}$$

$$O(\epsilon^3): \begin{cases} L^{(1)}\left(f^{(3)}, w^{(3)}\right) = (1+\xi)^2 H\left[S\left(w^{(1)}, \frac{1}{2} w^{(2)}\right) + S\left(w^{(2)}, \frac{1}{2} w^{(1)}\right) + S\left(w^{(2)}, \overline{w}\right)\right] & (5.8) \\ L^{(2)}\left(f^{(3)}, w^{(3)}\right) = -HK(\xi)\left[S\left(w^{(1)}, f,_x^{(2)}\right) + S\left(w^{(2)}, f^{(1)}\right) + S\left(\overline{w}, f^{(2)}\right)\right] & (5.9) \end{cases}$$

etc.

where,

$$w^{(i)} = w,_x^{(i)} = f^{(i)} = f,_x^{(i)} = 0 \quad \text{at} \quad x = 0, \pi, \quad i = 1,2,3, \ldots \tag{5.10}$$

5.1 SOLUTION OF EQUATION OF ORDER ϵ

For solution of (5.4) and (5.5) we let

$$\begin{pmatrix} w^{(1)} \\ f^{(1)} \end{pmatrix} = \sum_{r,p=1}^{\infty} \begin{pmatrix} w_{(r,p)}^{(1)} \\ f_{(r,p)}^{(1)} \end{pmatrix} (1 - \cos 2rx) \sin py \tag{5.11}$$

On assuming (5.11) we note that

$$\begin{pmatrix} \overline{\nabla}^4 w^{(1)} \\ \overline{\nabla}^4 f^{(1)} \end{pmatrix} = \sum_{r,p=1,2,3,\ldots}^{\infty} \begin{pmatrix} w_{(r,p)}^{(i)} \\ f_{(r,p)}^{(i)} \end{pmatrix} [(-16r^4 + 8\xi(rp)^2) \cos 2rx \sin py + (\xi p^2)^2 (1 - \cos 2rx) \sin py] \tag{5.12}$$

The aim is to choose r and p (positive integers) such that we have nontrivial values of $w^{(1)}$ and $f^{(1)}$. On substituting (5.12) into (5.4), multiplying after by $\cos 2mx \sin ny$ (for m, n fixed and positive integers), we see that for $r = m$, $p = n$, we integrate to get

$$f_{(m,n)}^{(1)} = \frac{-4(1+\xi)^2 m^2 w_{(m,n)}^{(1)}}{(4m^2 + n^2\xi)^2} \tag{5.13}$$

Next, we substitute (5.12) and (5.11) into (5.5) (assuming (5.2)), multiply by $\cos 2mx \sin ny$ and for $r = m, p = n$, we integrate to get

$$[(4m^2 + n^2\xi) - \lambda(2\alpha m^2 - n^2\xi)] w_{(m,n)}^{(1)} + 4m^2 K(\xi) f_{(m,n)}^{(1)} = \lambda(\bar{a} + 1)(2\alpha m^2 - n^2\xi)$$

On substituting for $f_{(m,n)}^{(1)}$ in (5.14) from (5.13) we simplify to get

$$w_{(m,n)}^{(1)} = \frac{\lambda(\bar{a}+1)(2\alpha m^2 - n^2\xi)}{(4m^2 + n^2\xi) + \left(\frac{4mA}{4m^2 + n^2\xi}\right)^2 - \lambda(2\alpha m^2 - n^2\xi)} \tag{5.15}$$

where we have substituted for $K(\xi)$. At this stage, we get

$$w^{(i)} = w_{(m,n)}^{(1)} (1 - \cos 2mx) \sin ny$$

$$f^{(1)} = -\Theta_0 w_{(m,n)}^{(1)} (1 - \cos 2mx) \sin ny, \quad \Theta_0 = \left(\frac{2m(1+\xi)}{4m^2 + n^2\xi}\right)^2 \tag{5.16}$$

5.2 SOLUTION OF EQUATION OF ORDER ϵ^2

On substituting (5.16) on the right hand sides of (5.6) and (5.7) and simplifying, we get

and
$$L^{(1)}\left(f^{(2)},w^{(2)}\right) = -4H(1+\xi)^2(mn)^2\left(w^{(1)^2}_{(m,n)} + 2\bar{a}w^{(1)}_{(m,n)}\right)$$
$$\times[\cos 2mx\sin^2 ny + \cos^2 2mx \sin^2 ny + \sin^2 2mx\cos^2 ny]$$
$$= -4H(1+\xi)^2(mn)^2\left(w^{(1)^2}_{(m,n)} + 2\bar{a}w^{(1)}_{(m,n)}\right)\Big[\frac{1}{2}\cos 2mx(1-\cos 2ny)$$
$$+\frac{1}{4}(1-\cos 2ny + \cos 4mx - \cos 4mx\cos 2ny)$$
$$+\frac{1}{4}(1+\cos 2ny - \cos 4mx - \cos 4mx\cos 2ny)\Big] \quad (5.17)$$

$$L^{(2)}\left(f^{(2)},w^{(2)}\right) = 8(mn)^2\Theta_0 HK(\xi)\left(w^{(1)^2}_{(m,n)} + \bar{a}w^{(1)}_{(m,n)}\right)$$
$$\times[\cos 2mx\sin^2 ny\,(1-\cos 2mx) + \sin^2 2mx\cos^2 ny]$$
$$= 8(mn)^2\Theta_0 HK(\xi)\left(w^{(1)^2}_{(m,n)} + 2\bar{a}w^{(1)}_{(m,n)}\right)\Big[\frac{1}{2}\cos 2mx(1-\cos 2ny)$$
$$+\frac{1}{4}(1+\cos 4mx - \cos 2ny - \cos 4mx\cos 2ny)$$
$$+\frac{1}{4}(1-\cos 4mx - \cos 2ny - \cos 4mx\cos 2ny)\Big] \quad (5.18)$$

It is clear from (5.17) and (5.18) that at this stage, there will be only two buckling modes generated by the terms $\cos 2mx\sin ny$ and $\cos 4mx\cos 2ny$. From this stage onwards, we shall assume

$$\begin{pmatrix} w^{(i)} \\ f^{(i)} \end{pmatrix} = \sum_{r,p=1,2,3..}^{\infty}\left[\left\{\begin{pmatrix} w^{(i)}_{1(r,p)} \\ f^{(i)}_{1(r,p)} \end{pmatrix}\cos py + \begin{pmatrix} w^{(i)}_{2(r,p)} \\ f^{(i)}_{2(r,p)} \end{pmatrix}\sin py\right\}(1-\cos 2rx)\right] \quad (5.19)$$

Using (5.19), we note that

$$\bar{\nabla}^4 w^{(i)} = \sum_{r,p=1,2,3..}^{\infty}\Big[w^{(i)}_{1(r,p)}\{-(16r^4 + 8\xi(rp)^2)\cos 2rx\cos py$$
$$+\xi^2p^4(1-\cos 2rx)\cos py\} + w^{(i)}_{2(r,p)}\{-(16r^4 + 8\xi(rp)^2)\cos 2rx\sin py$$
$$+\xi^2p^4(1-\cos 2rx)\cos py\}\Big] \quad (5.20a)$$

and

$$\bar{\nabla}^4 f^{(i)} = \sum_{r,p=1,2,3..}^{\infty}\Big[f^{(i)}_{1(r,p)}\{-(16r^4 + 8\xi(rp)^2)\cos 2rx\cos py$$
$$+\xi^2p^4(1-\cos 2rx)\cos py\} + f^{(i)}_{2(r,p)}\{-(16r^4 + 8\xi(rp)^2)\cos 2rx\sin py$$
$$+\xi^2p^4(1-\cos 2rx)\cos py\}\Big] \quad (5.20b)$$

Thus substituting (5.20a) into (5.17) and multiplying by $\cos 2mx\cos 2ny$ we see that for $r = m, p = 2n$, we get

$$f^{(2)}_{1(m,2m)} = -\Theta_1\left(w^{(i)^2}_{(m,n)} + 2\bar{a}w^{(i)}_{(m,n)}\right) - \Theta_2 w^{(2)}_{1(m,2n)} \quad (5.20c)$$

$$\Theta_1 = \frac{(mn)^2(1+\xi)^2H}{4(m^2+n^2\xi)^2}, \quad \Theta_2 = \frac{1}{4}\left(\frac{m(1+\xi)}{m^2+n^2\xi}\right)^2 \quad (5.20d)$$

Next, we multiply (5.17) by $\cos 4mx\cos 2ny$ and for $r = 2m, p = 2n$, we get

$$f^{(2)}_{1(2m,2m)} = -\Theta_3\left(w^{(i)^2}_{(m,n)} + 2\bar{a}w^{(i)}_{(m,n)}\right) - \Theta_4 w^{(2)}_{1(2m,2n)} \quad (5.21a)$$

$$\Theta_3 = \frac{(mn)^2(1+\xi)^2 H}{64(m^2+n^2\xi)^2}, \quad \Theta_4 = \left(\frac{4m(1+\xi)}{16m^2+4n^2\xi}\right)^2 \tag{5.21b}$$

We now substitute into (5.18) using (5.20a), multiply by $\cos 2mx \cos 2ny$ and for $r = m$, $p = 2n$, we simplify to get

$$w^{(2)}_{1(m,2n)}[16(m^2+n^2\xi)^2 - 2\lambda(\alpha m^2 - 2n^2\xi)] + 4m^2K(\xi)f^{(2)}_{1(m,2n)} = 4(mn)^2\Theta_0 K(\xi)\left(w^{(1)^2}_{(m,n)} + \bar{a}w^{(1)}_{(m,n)}\right). \tag{5.22a}$$

If we substitute for $f^{(2)}_{1(m,2n)}$ in (5.22a), we get

$$w^{(2)}_{1(m,2n)} = \frac{-\Theta_1\left(w^{(1)^2}_{(m,n)} + 2\bar{a}w^{(1)}_{(m,n)}\right) + 4(mn)^2\Theta_0 K(\xi)\left(w^{(1)^2}_{(m,n)} + \bar{a}w^{(1)}_{(m,n)}\right)}{16(m^2+n^2\xi)^2 + \left(\frac{2mA}{1+\xi}\right)^2\Theta_2 - 2\lambda(\alpha m^2 - 2n^2\xi)} \tag{5.22b}$$

Similarly, if we multiply (5.18) by $\cos 4mx \cos 2ny$, then for $r = 2m$, $p = 2n$, we simplify to get

$$w^{(2)}_{1(2m,2n)}[(16m^2+n^2\xi)^2 - 4\lambda(2m\alpha - n^2\xi)] + 16m^2K(\xi)f^{(2)}_{1(2m,2n)} = 4(mn)^2\Theta_0 K(\xi)\left(w^{(1)^2}_{(m,n)} + \bar{a}w^{(1)}_{(m,n)}\right) \tag{5.23a}$$

On substituting for $f^{(2)}_{1(2m,2n)}$, in (5.23a) we get

$$w^{(2)}_{1(2m,2n)} = \frac{K(\xi)\Theta_3\left(w^{(1)^2}_{(m,n)} + 2\bar{a}w^{(1)}_{(2m,2n)}\right) + 4(mn)^2\Theta_4 K(\xi)\left(w^{(1)^2}_{(m,n)} + \bar{a}w^{(1)}_{(m,n)}\right)}{(16m^2+4n^2\xi)^2 + \left(\frac{mA}{1+\xi}\right)^2\Theta_4 - 4\lambda(2\alpha m^2 - n^2\xi)} \tag{5.23b}$$

We can further write $w^{(2)}_{1(m,2n)}$ and $w^{(2)}_{1(2m,2n)}$ as

$$w^{(2)}_{1(m,2n)} = \Theta_5 w^{(1)^2}_{(m,n)} + \Theta_6\bar{a}w^{(1)}_{(m,n)} \tag{5.23c}$$

and

$$w^{(2)}_{1(2m,2n)} = \Theta_7 w^{(1)^2}_{(m,n)} + \Theta_8\bar{a}w^{(1)}_{(m,n)} \tag{5.23d}$$

where,

$$\Theta_5 = \frac{(2mnA)^2\Theta_0 HK(\xi) - \Theta_1}{16(m^2+n^2\xi)^2 + \left(\frac{mA}{1+\xi}\right)^2\Theta_2 - 2\lambda(\alpha m^2 - 2n^2\xi)} \tag{5.23e}$$

$$\Theta_6 = \frac{(2mnA)^2\Theta_0 HK(\xi) - 2\Theta_1}{16(m^2+n^2\xi)^2 + \left(\frac{2mA}{1+\xi}\right)^2\Theta_2 - 2\lambda(\alpha m^2 - 2n^2\xi)} \tag{5.23f}$$

$$\Theta_7 = \frac{K(\xi)\Theta_3 + (2mnA)^2\Theta_4 HK(\xi)\Theta_0}{16(m^2+n^2\xi)^2 + \left(\frac{2mA}{1+\xi}\right)^2\Theta_2 - 2\lambda(\alpha m^2 - 2n^2\xi)} \tag{5.23g}$$

$$\Theta_8 = \frac{-\left(\frac{2mA}{1+\xi}\right)^2\Theta_4 H - \left(\frac{A}{1+\xi}\right)^2\Theta_3}{16(m^2+n^2\xi)^2 + \left(\frac{2mA}{1+\xi}\right)^2\Theta_2 - 2\lambda(\alpha m^2 - 2n^2\xi)} \tag{5.23h}$$

Thus, at this order of perturbation, there are just two buckling modes with their corresponding Airy stress functions and these are

$$w^{(2)}_{1(m,2n)}(1 - \cos 2mx)\cos 2ny \quad \text{and} \quad w^{(2)}_{1(2m,2n)}(1 - \cos 4mx)\cos 2ny$$

with respective stress functions given as

$$f^{(2)}_{1(m,2n)}(1 - \cos 2mx)\cos 2ny \quad \text{and} \quad f^{(2)}_{1(2m,2n)}(1 - \cos 4mx)\cos 2ny$$

5.3 SOLUTION OF EQUATION OF ORDER ϵ^3

We now substitute on the right-hand sides (5.8) and (5.9) and simplify to get

$$L^{(1)}\left(f^{(3)},w^{(3)}\right)=-\frac{H(1+\xi)^2(mn)^2}{4}\left[\left\{20\left(w^{(1)}_{(m,n)}w^{(2)}_{1(m,2n)}+\bar{a}w^{(2)}_{(m,2n)}\right)(1-\cos 2mx\sin 3ny\right.\right.$$
$$-2\cos 2mx\sin ny-\cos 4mx\sin 3ny+\cos 4mx\sin ny)\}$$
$$+\left\{16\left(w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}+\bar{a}w^{(2)}_{1(2m,2n)}\right)(\cos 2mx\cos ny-\cos 2mx\cos 3ny\right.$$
$$-\cos 6mx\cos ny+\cos 6mx\cos 3ny)\}$$
$$+\left\{16\left(w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}+\bar{a}w^{(2)}_{1(2m,2n)}\right)(-\cos 2mx\cos ny+\cos 2mx\cos 3ny\right.$$
$$-\cos 6mx\cos ny+\cos 6mx\cos 3ny)\}$$
$$-\left\{32w^{(1)}_{(m,n)}w^{(2)}_{1(m,2n)}(\sin 3ny+\sin ny-\cos 4mx\sin 3ny-\cos 4mx\sin ny)\right\}$$
$$+\left\{32w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}(\cos 2mx\sin 3ny+\cos 2mx\cos 3ny\right.$$
$$\left.\left.-\cos 6mx\cos 3ny-\cos 6mx\cos ny)\right\}\right] \tag{5.24}$$

and

$$L^{(2)}\left(f^{(3)},w^{(3)}\right)=\frac{HK(\xi)(mn)^2}{4}\left[\left\{16\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)\right.\right.$$
$$\left.+4\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+8\bar{a}f^{(2)}_{1(m,2n)}\right\}(-1-2\cos 2mx\sin 3ny$$
$$-2\cos 2mx\sin ny-\cos 4mx\sin 3ny+\cos 4mx\sin ny)$$
$$+\left\{16\left(w^{(1)}_{(m,n)}f^{(2)}_{1(2m,2n)}+w^{(2)}_{1(2m,2n)}f^{(1)}_{(m,n)}\right)+4\bar{a}f^{(2)}_{1(2m,2n)}\right\}(\cos 2mx\,cos\,ny$$
$$-\cos 2mx\cos 3ny-\cos 6mx\cos ny+\cos 6mx\cos 3ny)$$
$$+\left\{16\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+\bar{a}f^{(2)}_{1(2m,2n)}\right\}(\sin 3ny+sin\,ny$$
$$-\cos 4mx\sin 3ny-\cos 4mx\sin ny)$$
$$+\left\{32w^{(1)}_{(m,n)}f^{(2)}_{1(2m,2n)}+w^{(2)}_{1(2m,2n)}f^{(1)}_{(m,n)}+\bar{a}f^{(2)}_{1(2m,2n)}\right\}$$
$$\times(\cos 2mx\cos 3ny+\cos 2mx\cos ny-\cos 6mx\cos 3ny$$
$$\left.-\cos 6mx\cos ny)\right] \tag{5.25}$$

From the simplifications on the right hand sides of (5.24) and (5.25), it is obvious that there will be eight distinct and nontrivial eigen buckling modes namely $w^{(3)}_{i(r,p)}$ with their respective Airy stress functions $f^{(3)}_{i(r,p)}$. These buckling modes correspond to the following terms on the right hand sides of (5.24) and (5.25):

$\cos 2mx\cos ny$, $\cos 2mx\sin ny$, $\cos 2mx\cos 3ny$, $\cos 2mx\sin 3ny$, $\cos 4mx\sin ny$, $\cos 4mx\sin 3ny$, $\cos 6mx\cos ny$ and $\cos 6mx\cos 3ny$.

Next, we substitute (5.20a,b) into the left hand side of (5.24) multiply by $\cos 2mx\cos ny$ and integrate and for $r=m$ and $p=n$, we get, after simplification

$$f^{(3)}_{1(m,n)}=\frac{8H(1+\xi)^2(mn)^2w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}-(1+\xi)^2m^2\,w^{(3)}_{1(m,n)}}{(4m^2+n^2\xi)^2} \tag{5.26a}$$

If we simplify (5.26a), we get

$$f^{(3)}_{1(m,n)}=\Theta_9w^{(1)}_{(m,n)}\left(\Theta_7{w^{(1)}_{(m,n)}}^2+\Theta_8\bar{a}w^{(1)}_{(m,n)}\right)-\Theta_{10}w^{(3)}_{1(m,n)} \tag{5.26b}$$

where,

$$\Theta_9 = \frac{8H(1+\xi)^2(mn)^2}{(4m^2+n^2\xi)^2}, \qquad \Theta_{10} = \frac{4(1+\xi)^2m^2}{(4m^2+n^2\xi)^2} \tag{5.26c}$$

Next, we multiply (5.24) by $\cos 2mx \sin ny$, integrate, and for $r = m, p = n$, we get

$$f_{2(m,n)}^{(3)} = -\frac{10(1+\xi)^2(mn)^2\left(w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}+\bar{a}w_{1(m,2n)}^{(2)}\right)-4m^2(1+\xi)^2\, w_{2(m,n)}^{(3)}}{(4m^2+n^2\xi)^2} \tag{5.27a}$$

On substituting for $w_{1(m,n)}^{(2)}$ in (5.27a), we get

$$f_{2(m,n)}^{(3)} = -\Theta_{11}\left(\Theta_5 w_{(m,n)}^{(1)^2} + \Theta_6\bar{a}w_{(m,n)}^{(1)}\right)\left(\bar{a}+w_{(m,n)}^{(1)}\right) - \Theta_{10}w_{2(m,n)}^{(3)} \tag{5.27b}$$

where,

$$\Theta_{11} = \frac{10Hm^2(1+\xi)^2}{(4m^2+n^2\xi)^2} \tag{5.27c}$$

Next, we multiply (5.24) by $\cos 2mx \cos 3ny$, integrate, and for $r = m$, $p = 3n$, we get

$$f_{1(m,n)}^{(3)} = \frac{8H(1+\xi)^2(mn)^2w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}-4m^2(1+\xi)^2\, w_{1(m,3n)}^{(3)}}{(4m^2+9n^2\xi)^2} \tag{5.28a}$$

On substituting for $w_{1(2m,2n)}^{(2)}$ in (5.28a), and simplifying, we get

$$f_{1(m,3m)}^{(3)} = \Theta_{12}w_{(m,n)}^{(1)}\left(\Theta_7 w_{(m,n)}^{(1)^2} + \Theta_8 w_{(m,n)}^{(1)}\right) - \Theta_{13}w_{1(m,3n)}^{(3)} \tag{5.28b}$$

where,

$$\Theta_{12} = \frac{8H(1+\xi)^2(mn)^2}{(4m^2+9n^2\xi)^2}, \qquad \Theta_{13} = \frac{4(1+\xi)^2m^2}{(4m^2+9n^2\xi)^2} \tag{5.28c}$$

We next multiply (5.24) by $\cos 2mx \sin 3ny$, integrate, and for $r = m$, $p = 3n$, we get

$$f_{2(m,3n)}^{(3)} = \frac{-5H(1+\xi)^2(mn)^2\left(w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}+\bar{a}w_{1(m,2n)}^{(2)}\right)-4m^2(1+\xi)^2\, w_{2(m,3n)}^{(3)}}{(4m^2+9n^2\xi)^2} \tag{5.29a}$$

On substituting for $w_{1(m,2n)}^{(2)}$ in (5.29a), we get

$$f_{2(m,3m)}^{(3)} = \Theta_{14}\left(\Theta_5 w_{(m,n)}^{(1)^2} + \Theta_6\bar{a}w_{(m,n)}^{(1)}\right)\left(\bar{a}+w_{(m,n)}^{(1)}\right) - \Theta_{13}w_{2(m,3n)}^{(3)} \tag{5.29b}$$

where,

$$\Theta_{14} = \frac{-5H(1+\xi)^2(mn)^2}{(4m^2+9n^2\xi)^2} \tag{5.29c}$$

Next, we multiply (5.24) by $\cos 4mx \sin ny$, integrate, and for $r = 2m$, $p = n$, we get

$$f_{2(2m,n)}^{(3)} = \frac{H(1+\xi)^2(mn)^2\left\{5\left(w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}+\bar{a}w_{1(2m,2n)}^{(2)}\right)+8w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}\right\}-16m^2(1+\xi)^2\, w_{2(2m,n)}^{(3)}}{(16m^2+n^2\xi)^2} \tag{5.30a}$$

On substituting for $w_{1(m,2n)}^{(2)}$ in (5.30a), we get

$$f_{2(2m,n)}^{(3)} = \Theta_{15}\left[5\left(\Theta_5 w_{(m,n)}^{(1)^2} + \Theta_6\bar{a}w_{(m,n)}^{(1)}\right)\left(\bar{a}+w_{(m,n)}^{(1)} + 8\right)\right] - \Theta_{16}w_{2(2m,n)}^{(3)} \tag{5.30b}$$

where,

$$\Theta_{15} = \frac{H(1+\xi)^2(mn)^2}{(16m^2+9n^2\xi)^2}, \qquad \Theta_{16} = \frac{16(1+\xi)^2m^2}{(16m^2+9n^2\xi)^2} \tag{5.30c}$$

On multiplying (5.24) by $\cos 4mx \sin 3ny$ and integrating so that for $r = 2m, p = 3n$, we get

$$f_{2(2m,3n)}^{(3)} = \frac{-5H(1+\xi)^2(mn)^2\left(w_{(m,n)}^{(1)}w_{1(m,2n)}^{(2)}+\bar{a}w_{1(m,2n)}^{(2)}\right)-16m^2(1+\xi)^2\, w_{2(2m,3n)}^{(3)}}{(16m^2+9n^2\xi)^2} \tag{5.31a}$$

On simplifying (5.31a), we get

$$f^{(3)}_{2(2m,3m)} = \Theta_{17}\left(\Theta_{15}w^{(1)^2}_{(m,n)} + \Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+w^{(1)}_{(m,n)}\right) - \Theta_{16}w^{(3)}_{2(2m,3n)} \quad (5.31b)$$

where,

$$\Theta_{17} = -\frac{5H(1+\xi)^2(mn)^2}{(16m^2+9n^2\xi)^2} \quad (5.31c)$$

Next, we multiply (5.24) by $\cos 6mx \cos ny$, integrate, and for $r = 3m$, $p = n$, we get

$$f^{(3)}_{(3m,n)} = -\left[\frac{8H(1+\xi)^2(mn)^2\left\{\left(w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}+\bar{a}w^{(2)}_{1(2m,2n)}\right)-w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}\right\}+36m^2(1+\xi)^2\, w^{(3)}_{1(3m,n)}}{(36m^2+n^2\xi)^2}\right] \quad (5.32a)$$

After substituting for $w^{(2)}_{1(2m,2n)}$ in (5.32a), we get

$$f^{(3)}_{1(3m,m)} = -\Theta_{18}\left[\left(\Theta_7w^{(1)^2}_{(m,n)} + \Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+2w^{(1)}_{(m,n)}\right)\right] - \Theta_{19}w^{(3)}_{1(3m,n)} \quad (5.32b)$$

where,

$$\Theta_{18} = \frac{8H(1+\xi)^2(mn)^2}{(36m^2+n^2\xi)^2}, \qquad \Theta_{19} = \frac{36(1+\xi)^2m^2}{(36m^2+n^2\xi)^2} \quad (5.32c)$$

Lastly, we multiply (5.24) by $\cos 6mx \cos 3ny$ and for $r = 3m$, $p = 3n$, we get

$$f^{(3)}_{1(3m,3n)} = \frac{H(1+\xi)^2(mn)^2\left\{4\left(w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}+\bar{a}w^{(2)}_{1(2m,2n)}\right)-8w^{(1)}_{(m,n)}w^{(2)}_{1(2m,2n)}\right\}-36m^2(1+\xi)^2\, w^{(3)}_{1(3m,3n)}}{(36m^2+9n^2\xi)^2} \quad (5.33a)$$

On substituting for $w^{(2)}_{1(2m,2n)}$ in (5.33a), we get

$$f^{(3)}_{1(3m,3m)} = 4\Theta_{20}\left(\Theta_7w^{(1)^2}_{(m,n)} + \Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}-w^{(1)}_{(m,n)}\right) - \Theta_{21}w^{(3)}_{1(3m,3n)} \quad (5.33b)$$

where,

$$\Theta_{20} = \frac{H(1+\xi)^2(mn)^2}{(36m^2+9n^2\xi)^2}, \qquad \Theta_{21} = \frac{36(1+\xi)^2m^2}{(36m^2+9n^2\xi)^2} \quad (5.33c)$$

Next, we substitute on the left hand side of (5.25) using (5.20a), thereafter multiply by $\cos 2mx \cos ny$ and note that for $r = m, p = n$, we get, (without further simplification)

$$w^{(3)}_{1(m,n)} = \left[\frac{\left(\frac{2mA}{1+\xi}\right)^2\left\{\Theta_9w^{(1)}_{(m,n)}\left(\Theta_7w^{(1)^2}_{(m,n)}+\Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\right\}+H\left(\frac{mnA}{1+\xi}\right)^2\left\{H\left(\frac{mnA}{1+\xi}\right)^2\left[12\left(w^{(1)}_{(m,n)}f^{(2)}_{1(2m,n)}+w^{(2)}_{1(2m,2n)}f^{(1)}_{(m,n)}\right)+9\bar{a}f^{(2)}_{1(2m,2n)}\right]\right\}}{(4m^2+n^2\xi)^2+\Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-n^2\xi)}\right] \quad (5.34)$$

We can easily simplify (5.34) by substituting the relevant terms there.

On multiplying (5.25) by $\cos 2mx \sin ny$, we get

$$w^{(3)}_{2(m,n)} = \left[\frac{\frac{HK(\xi)(mn)^2}{2}\left[16\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+4\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+8\bar{a}f^{(2)}_{1(m,2n)}\right]+\left(\frac{2mA}{1+\xi}\right)^2\Theta_{11}\left(\Theta_5w^{(1)}_{(m,n)}+\Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+w^{(1)}_{(m,n)}\right)}{4(m^2+n^2\xi)^2+\Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-n^2\xi)}\right] \quad (5.35a)$$

Further simplification of (5.35a) yields

$$w^{(3)}_{2(m,n)} = \left[\frac{\frac{-HK(\xi)(mn)^2}{2}\left[20w^{(1)^3}_{(m,n)}(\Theta_0\Theta_5-\Theta_1-\Theta_2\Theta_5)+w^{(1)^2}_{(m,n)}\{20(\Theta_0\Theta_6\bar{a}+\Theta_0\Theta_2\bar{a}-2\bar{a}\Theta_1)+8\bar{a}(\Theta_1-\Theta_2\Theta_5)\}+8\bar{a}w^{(1)}_{(m,n)}(2\Theta_1\bar{a}-\Theta_2\Theta_6\bar{a})\right]+\left(\frac{2mA}{1+\xi}\right)^2\Theta_{11}\left(\Theta_5w^{(1)}_{(m,n)}+\Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+w^{(1)}_{(m,n)}\right)}{(4m^2+n^2\xi)^2+\Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-n^2\xi)}\right] \quad (5.35b)$$

Next, we multiply (5.25) by $\cos 2mx \cos 3ny$, and for $r = m, p = 3n$, we get

$$w^{(3)}_{1(m,3n)} = \left[\frac{\begin{array}{c} HK(\xi)(mn)^2\left[4\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+7\bar{a}f^{(2)}_{1(m,2n)}\right] \\ -\left(\frac{2mA}{1+\xi}\right)^2\left\{\Theta_{12}w^{(1)}_{(m,n)}\left(\Theta_7 w^{(1)^2}_{(m,n)}+\Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\right\}\end{array}}{4(m^2+9n^2\xi)^2+\Theta_{13}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-9n^2\xi)}\right] \quad (5.36)$$

Next, we multiply (5.25) by $\cos mx \sin 3ny$, and for $r = m$, $p = 3n$, we get

$$w^{(3)}_{2(m,3n)} = \left[\frac{\begin{array}{c} HK(\xi)(mn)^2\left[10\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+4\bar{a}f^{(2)}_{1(m,2n)}\right] \\ -\left(\frac{2mA}{1+\xi}\right)^2\Theta_{14}\left\{\left(\Theta_5 w^{(1)^2}_{(m,n)}+\Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+w^{(1)}_{(m,n)}\right)\right\}\end{array}}{4(m^2+9n^2\xi)^2+\Theta_{13}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-9n^2\xi)}\right] \quad (5.37)$$

Next, we multiply (5.25) by $\cos 4mx \sin ny$, and for $r = 2m$, $p = n$, we get

$$w^{(3)}_{2(2m,n)} = \left[\frac{\begin{array}{c} HK(\xi)(mn)^2\left[5\left(w^{(1)}_{(m,n)}f^{(2)}_{1(m,2n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+2\bar{a}f^{(2)}_{1(m,2n)}\right] \\ -\left(\frac{2mA}{1+\xi}\right)^2\left[\Theta_{15}\left\{\left(5\Theta_5 w^{(1)^2}_{(m,n)}+\Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+8+w^{(1)}_{(m,n)}\right)\right\}\right]\end{array}}{(16m^2+n^2\xi)^2+\Theta_{16}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(16\alpha m^2-n^2\xi)}\right] \quad (5.38)$$

Next, we multiply (5.25) by $\cos 4mx \sin 3ny$, and for $r = 2m$, $p = 3n$, we get

$$w^{(3)}_{2(2m,3n)} = \left[\frac{\begin{array}{c} \frac{HK(\xi)(mn)^2}{4}\left[32\left(w^{(1)}_{(m,n)}f^{(2)}_{1(2m,n)}+w^{(2)}_{1(m,2n)}f^{(1)}_{(m,n)}\right)+5\bar{a}f^{(2)}_{1(m,2n)}\right] \\ -\left(\frac{2mA}{1+\xi}\right)^2\left\{\Theta_{17}\left(\Theta_{15} w^{(1)^2}_{(m,n)}+\Theta_6\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+w^{(1)}_{(m,n)}\right)\right\}\end{array}}{(16m^2+9n^2\xi)^2+\Theta_{16}\left(\frac{4mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-9n^2\xi)}\right] \quad (5.39)$$

Next, we multiply (5.25) by $\cos 6mx \cos ny$, and for $r = 3m$, $p = n$, we get

$$w^{(3)}_{1(3m,n)} = \left[\frac{\begin{array}{c} HK(\xi)(mn)^2\left[12\left(w^{(1)}_{(m,n)}f^{(2)}_{1(2m,2n)}+w^{(2)}_{1(2m,2n)}f^{(1)}_{(m,n)}\right)+9\bar{a}f^{(2)}_{1(2m,2n)}\right] \\ +\left(\frac{3mA}{1+\xi}\right)^2\left\{\Theta_{18}\left(\Theta_7 w^{(1)^2}_{(m,n)}+\Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}+2w^{(1)}_{(m,n)}\right)\right\}\end{array}}{(36m^2+n^2\xi)^2+\Theta_{19}\left(\frac{3mA}{1+\xi}\right)^2-\lambda\left(\frac{9\alpha m^2}{2}-n^2\xi\right)}\right] \quad (5.40)$$

Lastly, we multiply (5.25) by $\cos 6mx \cos 3ny$, and for $r = 3m$, $p = 3n$, we get

$$w^{(3)}_{2(2m,n)} = \left[\frac{\begin{array}{c} HK(\xi)(mn)^2\left[4\left(w^{(1)}_{(m,n)}f^{(2)}_{1(2m,3n)}+w^{(2)}_{1(2m,2n)}f^{(1)}_{(m,n)}\right)+8\bar{a}f^{(1)}_{(m,n)}\right] \\ -\left(\frac{3mA}{1+\xi}\right)^2\left\{4\Theta_{20}\left(\Theta_7 w^{(1)^2}_{(m,n)}+\Theta_8\bar{a}w^{(1)}_{(m,n)}\right)\left(\bar{a}-w^{(1)}_{(m,n)}\right)\right\}\end{array}}{(36m^2+9n^2\xi)^2+\Theta_{21}\left(\frac{3mA}{1+\xi}\right)^2-\lambda\left(\frac{9\alpha m^2}{2}-9n^2\xi\right)}\right] \quad (5.41)$$

As a summary so far, we can write the displacement (i.e. eigen buckling modes) and the respective Airy stress function as

$$\begin{pmatrix} w \\ f \end{pmatrix} = \epsilon \begin{pmatrix} w^{(1)}_{(m,n)} \\ f^{(1)}_{(m,n)} \end{pmatrix}(1-\cos 2mx)\sin ny$$
$$+\epsilon^2\left[\begin{pmatrix} w^{(2)}_{1(m,n)} \\ f^{(2)}_{1(m,n)} \end{pmatrix}(1-\cos 2mx)\cos ny + \begin{pmatrix} w^{(2)}_{1(m,2n)} \\ f^{(2)}_{1(m,2n)} \end{pmatrix}(1-\cos 2mx)\cos 2ny\right]$$

$$+\epsilon^3\left[\left\{\begin{pmatrix} w_{1(m,n)}^{(3)} \\ f_{1(m,n)}^{(3)} \end{pmatrix}\cos ny+\begin{pmatrix} w_{2(m,n)}^{(2)} \\ f_{2(m,n)}^{(2)} \end{pmatrix}\sin ny\right\}(1-\cos 2mx)+\left\{\begin{pmatrix} w_{1(m,3n)}^{(3)} \\ f_{1(m,3n)}^{(3)} \end{pmatrix}\cos 3ny\right.\right.$$
$$\left.+\begin{pmatrix} w_{2(m,3n)}^{(3)} \\ f_{2(m,3n)}^{(3)} \end{pmatrix}\sin 3ny\right\}(1-\cos 2mx)+\begin{pmatrix} w_{2(2m,n)}^{(3)} \\ f_{2(2m,n)}^{(3)} \end{pmatrix}(1-\cos 4mx)\sin ny$$
$$+\begin{pmatrix} w_{2(2m,3n)}^{(3)} \\ f_{2(2m,3n)}^{(3)} \end{pmatrix}(1-\cos 4mx)\sin 3ny+\begin{pmatrix} w_{1(3m,n)}^{(3)} \\ f_{1(3m,n)}^{(3)} \end{pmatrix}(1-\cos 6mx)\cos ny$$
$$\left.+\begin{pmatrix} w_{1(3m,3n)}^{(3)} \\ f_{1(3m,3n)}^{(3)} \end{pmatrix}(1-\cos 6mx)\cos 3ny\right]+\cdots \qquad (5.42)$$

A diagrammatic representation of the eigen buckling modes can be seen in Figure below:

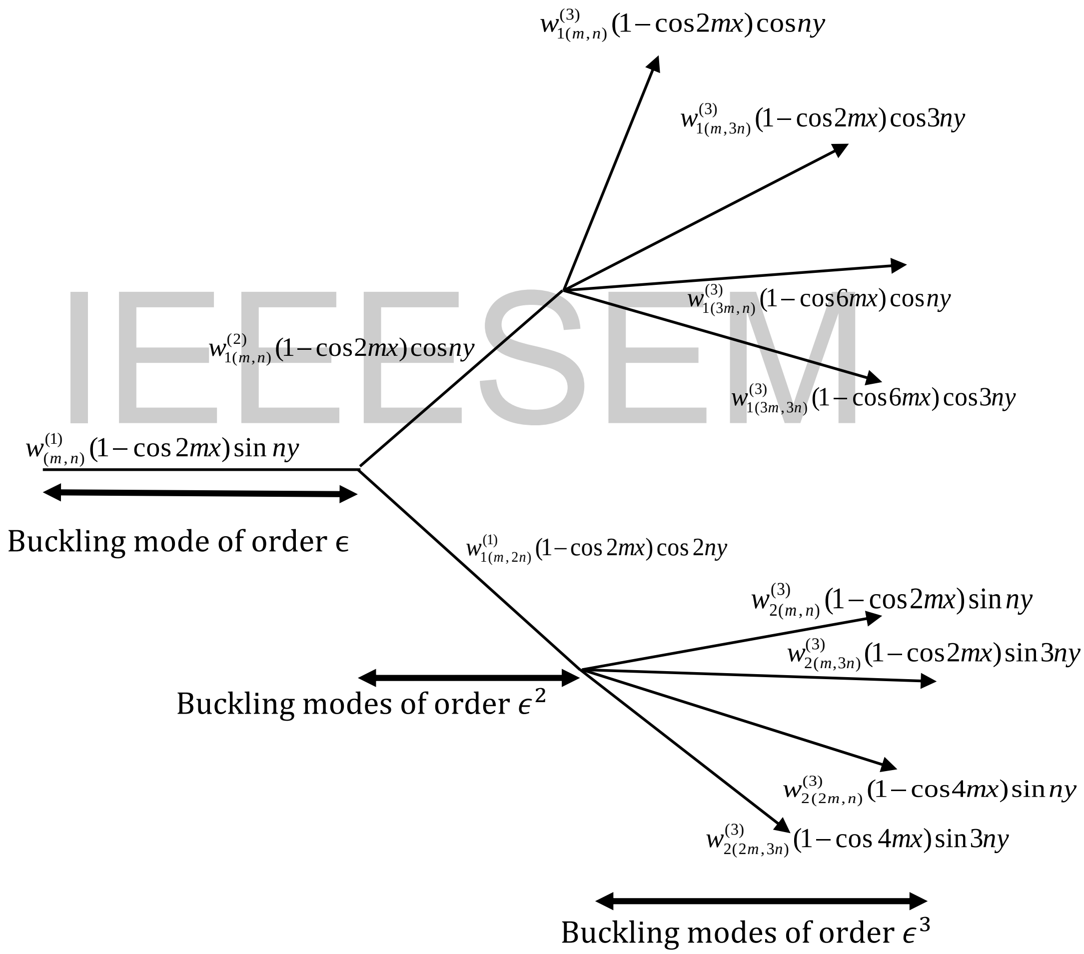

Figure 1: **The Bifurcation "Tree"**

Showing a diagrammatic representation of the splitting of eigen buckling modes at various orders of perturbations and various degrees of nonlinearities.

6. STATIC BUCKLING LOAD

The static buckling load λ_S [9] is determined from the maximization

$$\frac{d\lambda}{dw} = 0 \tag{6.1}$$

for the displacement as per equation (5.42). However, the analysis is significantly simplified if we take only the buckling modes in the shape of the imperfection. In this respect, the displacement becomes

$$w = \epsilon w^{(1)}_{(m,n)}(1-\cos 2mx)\sin ny + \epsilon^3 w^{(3)}_{2(m,n)}(1-\cos 2mx)\sin ny + \cdots \tag{6.2}$$

We note form (6.2) that terms of order ϵ^2 do not contribute in this case.

We shall next determine (6.1) at the critical values of x and y where w has a maximum value. These critical values are $x_a = \frac{\pi}{2m}$, $y_a = \frac{\pi}{2n}$, where x_a and y_a are the critical values of x and y respectively. The value of w at these values is

$$w_a = 2\epsilon w^{(1)}_{(m,n)} + 2\epsilon^3 w^{(3)}_{2(m,n)} + \cdots \tag{6.3}$$

As in [9], we shall reverse the series in (6.3) and so we see

$$\epsilon = d_1 w_a + d_3 w_a^3 + \cdots \tag{6.4a}$$

Meanwhile, we set

$$w_a = c_1\epsilon + c_3\epsilon^3 + \cdots \tag{6.4b}$$

where,

$$c_1 = 2w^{(1)}_{(m,n)}, \quad c_3 = 2w^{(3)}_{2(m,n)} \tag{6.4c}$$

By substituting (6.4b) into (6.1) and equating the coefficients of powers of orders of ϵ, we get

$$d_1 = \frac{1}{c_1}, \quad d_3 = -\frac{c_3}{{c_1}^4} \tag{6.5a}$$

The maximization (6.1) is now easily executed from (6.4a), where w_a is now being substituted for w and this yields

$$d_1 + 3d_3 w_{a_S}^2 = 0 \tag{6.5b}$$

where $w_{a_S}^2$ is the value of w_a^2 at static buckling and

$$w_{a_S}^2 = \frac{d_1}{3d_3} \tag{6.5c}$$

On substituting for d_1 and d_3 in (6.5c) and (6.5a), we get

$$w_{a_S} = \sqrt{\frac{C_1^3}{3C_3}} \tag{6.6}$$

where we have taken the positive value of w_{a_S}. The static buckling load λ_S, is determined by determining (6.4a) at static buckling load, where

$$\epsilon = w_{a_S}[d_1 + d_3 w_a^3] + \cdots \tag{6.7a}$$

This yields

$$\epsilon = \frac{2}{3\sqrt{3}}\left(\frac{C_1}{C_3}\right)^{\frac{1}{2}} \tag{6.7b}$$

Now, the component of C_3 coming from the buckling modes that are in the shape of imperfection is from (5.35b) and this yields

$$w^{(3)}_{2(m,n)} = \frac{\frac{H}{2}\left(\frac{2mA}{1+\xi}\right)^2\left[20 w^{(1)^3}_{(m,n)}(\Theta_1+\Theta_2\Theta_5-\Theta_0\Theta_3)\right]}{(4m^2+n^2\xi)^2+\Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-n^2\xi)} \qquad (6.8a)$$

where, we have substituted for $K(\xi)$. We can further simplify (6.8a) as

$$w^{(3)}_{2(m,n)} = \frac{\Theta_{22} w^{(1)^3}_{(m,n)}}{(4m^2+n^2\xi)^2+\Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2-\lambda(2\alpha m^2-n^2\xi)} \qquad (6.8b)$$

where,

$$\Theta_{22} = 10H\left(\frac{mA}{1+\xi}\right)^2(\Theta_1+\Theta_2\Theta_5-\Theta_0\Theta_3) \qquad (6.8c)$$

and where we have substituted for $K(\xi)$.
On substituting for terms into (6.7b), we get

$$\epsilon = \frac{2}{3\sqrt{3}\, w^{(1)}_{(m,n)}}\left(\frac{B}{\Theta_{22}}\right)^{\frac{1}{2}} \qquad (6.9a)$$

where,

$$B = (4m^2+n^2\xi)^2 + \Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2 - \lambda(2\alpha m^2 - n^2\xi) \qquad (6.9b)$$

We recall that

$$\Theta_{10} = \Theta_0 = \left(\frac{2m(1+\xi)}{4m^2+n^2\xi}\right)^2 \qquad (6.9c)$$

On substituting for $w^{(1)}_{(m,n)}$ in (6.9a), we get

$$\left[(4m^2+n^2\xi)^2 + \Theta_{10}\left(\frac{2mA}{1+\xi}\right)^2 - \lambda_S(2\alpha m^2 - n^2\xi)\right]^{\frac{3}{2}} = \frac{3\sqrt{3}}{2\Theta_{22}^{\frac{1}{2}}}\epsilon\lambda_S(\bar{a}+1)(2\alpha m^2 - n^2\xi) \qquad (6.10)$$

which gives an implicit formula for determining the static buckling load λ_S. Hence, $\Theta_{22}^{\frac{1}{2}}$ serves as the imperfection – sensitivity parameter and is such that if $\Theta_{22}^{\frac{1}{2}} > 0$ the structure is imperfection – sensitive whereas for $\Theta_{22}^{\frac{1}{2}} < 0$, the structure is imperfection – insensitive

7. DISCUSSION OF RESULTS

We observe from the results so far that the number of buckling modes generated at any stage depends on the order of perturbation involved or on the degree of the nonlinearity of the equations solved. Thus, at the linear stage (that is equations of order ϵ), there is only one buckling mode, namely $w^{(1)}_{(m,n)}$, whereas at the level of quadratic nonlinearity of the equations, (that is equations of order ϵ^2) there are two eigen buckling modes. Similarly, at the level of cubic nonlinearity (that is equations of order ϵ^2), there are eight Eigen buckling modes. We expect the number of buckling modes to increase further as the degree of nonlinearity of equations solved increases, that is, at a higher level perturbations.
The result in (6.10) is characteristics of all cubic structures [3] of which cylindrical shells are typical examples.

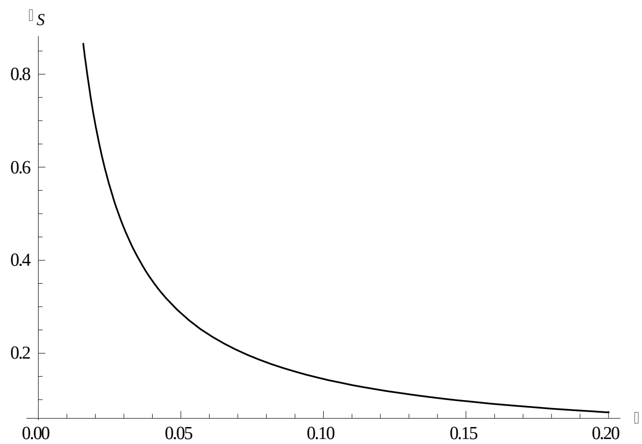

Figure2: **The graph of static buckling load λ_s against imperfection ϵ, where m=1, n=1, α=1, A=0.02, H=0.02, $\bar{a} = 0.002$, and ξ=0.8 .**

8. CONCLUSION

We have presented an analytical approach in solving a problem that at the best of time, is normally solved or treated numerically. We have tacitly assumed that the magnitude of imperfection is small compared to small thickness. However, the complexity of the perturbation procedure increases with increased order of perturbation and nonlinearity of the problem.

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