NUMERICAL SOLUTION OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH FINITE DEFERENCE METHODS

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Abstract

In this paper the finite difference approximation for hyperbolic partial differential equations was applied and both the explicit and implicit methods of finite difference approximations were discussed. As we have seen from the computation results, finite difference method of solving differential equations is mesh size dependent. That is the method of the accuracy increases when the mesh size is small enough. The computation result also indicates that, using implicit finite difference method to solve hyperbolic partial differential equations gives a better approximation than explicit finite difference approaches.

Keywords: - finite difference; Numerical methods; hyperbolic equations

1. Introduction

Differential equations are interesting and important because they express relationships involving rates of change. Such relationships form the basis for developing ideas and studying phenomena in the sciences, engineering, economics, and increasingly in other areas, such as the business world and the stock market. We will see examples of applications as we learn more about differential equations(O'Neil, 2007)&(Hoffman, 2001a). Although mathematics has been used for centuries in one form or another with in many areas of science and industry, a modern scientific computing using electronic computer has its origin in research and development during the Second World War. In the late forties and early fifties, the foundation of numerical analysis was laid as a separate discipline of mathematics.

Mathematical models based on partial differential equations (PDEs) are ubiquitous these days, arising in all areas of science and engineering, and also in medicine and finance. In many situations, finding the analytic solution to these partial differential equations or system of such equations is unrealistic or even impossible in very simple cases(Press, 1996). Therefore, numerical methods for finding approximate solutions to PDE problems are of great importance.

Recent modern development has increased enormously the scope for using numerical method. Not only has this been caused by the continuing advent of faster computer with large memories. Giant in problem solving capabilities through better mathematical algorithms have in many cases played an equally important role, this has meant that today one can treat much more complex and less simplified problems through massive amount of numerical calculation(Conte & De Boor, 1980). This development has caused the always close interaction between mathematics in one hand and science and technology on the other(Hoffman, 2001b). The focus of this paper is to find the numerical solution of partial differential equations by considering the important class of partial differential equations called **hyperbolic equations** by using finite difference method.

2. Definition and examples of partial differential equations

Definition 1: An equation containing the derivative or differential of one or more dependent variable with respect to one or more independent variable is called a differential equation (DE).

Example.1

a. $y''' - 4xy' + 7y = -2x$ and

b. $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 u}{\partial t^2}$

Definition 2: A partial differential equation (PDE) is an equation that involves one or more derivatives of a dependent with respect to two or more independent variables.

The following is an example of partial differential equation

$$A(t,x,y)U_t + B(t,x,y)U_x + C(t,x,y)U_y = \mathrm{F(t,x,y)} \tag{1.1}$$

Where: -

- t, x, y are the independent variables (often time and space)
- A, B, C and F are known functions of the independent variables,
- U is the dependent variable and is an unknown function of the independent variables.
- partial derivatives are denoted by subscripts: $U_t = \frac{\partial u}{\partial t}$, $U_x = \frac{\partial u}{\partial x}$, $U_{yy} = \frac{\partial^2 u}{\partial y^2}$

Definition 3: The order of a partial differential equation is the order of the highest derivative in the equation.

Example.2: a. $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 u}{\partial t^2}$ is second order partial differential equation.

b. $c^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$ is fourth order partial differential equation

Definition 4: A partial differential equation is said to be linear, when the dependent variable and its derivatives occur only in the first degree and no products of the dependent variable and its derivatives.

3. Classification of linear second order partial differential equations

The general linear differential equations of second order in two independent variables is of the form

$$\mathrm{A}U_{xx} + \mathrm{B}U_{xy} + \mathrm{C}U_{yy} + DU_x + \mathrm{E}U_y + \mathrm{FU} + \mathrm{G} = 0 \qquad (1.2)$$

Where A, B, C, D, E, F and G are functions of x and y

The partial differential equations is called Elliptic equations if $B^2 + 4AC < 0$

Parabolic equation if $B^2 + 4AC = 0$

Hyperbolic equation if $B^2 + 4AC > 0$

4. Finite Difference methods

Before addressing boundary value problems, it is better to develop further the notation of finite difference approximation of derivatives. Finite difference method for solving a partial differential equation can be done by transforming calculus problems in to an algebraic problem by

i. By discretizing the continuous physical domain in to discrete difference grids.
ii. Approximate the individual partial derivatives in the partial differential equation finite difference approximation.

iii. Substitute the finite differences in to the partial differential equations to obtain algebraic equations

iv. Solve the resulting algebraic partial differential equations.

Consider a function y(x) for which we want to compute the derivative y'(x) at some point x.

From Taylor's series expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \cdots \quad (1.3)$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2 y''(x)}{2!} - \cdots \quad (1.4)$$

From (3) we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} + o(h) \quad (1.5)$$

This is called the forward difference approximation

From (4) we have

$$y'(x) = \frac{y(x) - y(x-h)}{h} + o(h) \quad (1.6)$$

This is called the backward difference approximation

From (3) and (4) we have

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} + o(h^2) \quad (1.7)$$

This is called the central difference approximation for first order derivatives

Adding (3) and (4)

$$y(x+h) + y(x-h) = 2y(x) + 2\frac{h^2}{2!}y''(x) + 2\frac{h^4}{4!}y^{iv}(x) + \cdots$$

Truncating order of h^4 and above we, have

$$y''(x) \approx y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + o\left(h^2\right) \quad (1.8)$$

Mesh generation: suppose the region $0 \leq x \leq L$, $t>0$ be rectangular network of mesh lines. Let the interval [0, l] be divided in to M parts. Then the mesh length along the x-axis is $h = \frac{L}{M}$. The points along the x-axis are $x_i = ih$, i=0, 1, 2… M. Let the mesh length along the t-axis be k and define $t_j = jk$. The mesh points are (x_i, t_j). We call t_j as the j^{th} time level. At any point (x_i, t_j) we denote the numerical solution by$u_{i,j}$.

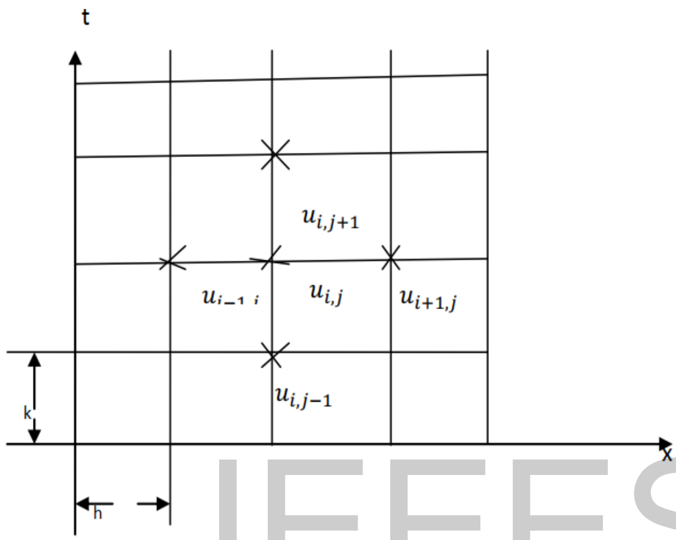

Figure 1: Grid points

By using the above coordinate plan

$$u_x \approx \frac{u_{i+1,j}-u_{i,j}}{h} + o(h) \qquad \text{Forward difference}$$

$$u_x \approx \frac{u_{i,j}-u_{i-1,j}}{h} + o(h) \qquad \text{Backward difference}$$

$$u_x \approx \frac{u_{i+1,j}-u_{i-1,j}}{h} + o(h) \qquad \text{Central difference}$$

$$u_{xx} \approx \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} + o(h^2)$$

Similarly, with respect to the independent variable t, we have

$$u_t \approx \frac{u_{i+1,j}-u_{i,j}}{h} + o(k)$$

$$u_t \approx \frac{u_{i,j}-u_{i-1,j}}{h} + o(k)$$

$$u_t \approx \frac{u_{i+1,j} - u_{i-1,j}}{h} + o(k)$$

$$u_{tt} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(k^2)$$

5. Solution of hyperbolic partial differential equations

5.1 Introduction about hyperbolic partial differential equations

In introduction section, we define the linear second order partial differential equation

$$\mathrm{A}U_{xx} + \mathrm{B}U_{xy} + \mathrm{C}U_{yy} + DU_x + \mathrm{E}U_y + \mathrm{FU} + \mathrm{G} = 0$$

and hyperbolic equation if $B^2 - 4AC > 0$. The simplest example of a hyperbolic equation is the one-dimensional wave equation. Study of the behavior of waves is one of the important areas in engineering. All vibration problems are governed by wave equations, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, t > 0, 0 \leq x \leq L$

Consider the problem of a vibrating elastic string of length L, located on the x-axis on the interval [0, L]. Let u(x, t) denote the displacement of the string in the vertical plane which is also the solution. Then, the vibration of the elastic string is governed by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, t > 0, \ 0 \leq x \leq L \qquad (2.1)$$

Where c^2 is a constant and depend on the material property of the string, the tension T in the string and the mass per unit length of the string. In order that the solution of the problem exists and unique, we need to prescribe the following conditions

i. Initial condition: Displacement at t=0 or initial displacement is given by

$$u(x,0) = f(x), 0 \leq x \leq L$$

Initial velocity: $u_t(x,0) = g(x)$, $0 \leq x \leq L$

ii. Boundary conditions: We consider the case when the ends of the string are fixed. Since the ends are fixed, we have the boundary conditions as u (0, t) =0, u (l, t) =0, t>0

5.2 Explicit method

Using the central difference method, we can write the approximation as

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \tag{2.2}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] \tag{2.3}$$

Next substituting (2.1) and (2.2) from (2.3) we can get

$$\frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] = \frac{c^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

*O*r $\quad u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \alpha\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right], \quad$ where $\alpha = \frac{c^2k^2}{h^2}$

Or $\quad u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \alpha\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$

Or $\quad u_{i,j+1} = 2(1-\alpha)u_{i,j} + \alpha\left[u_{i+1,j} + u_{i-1,j}\right] - u_{i,j-1} \quad (2.4)$

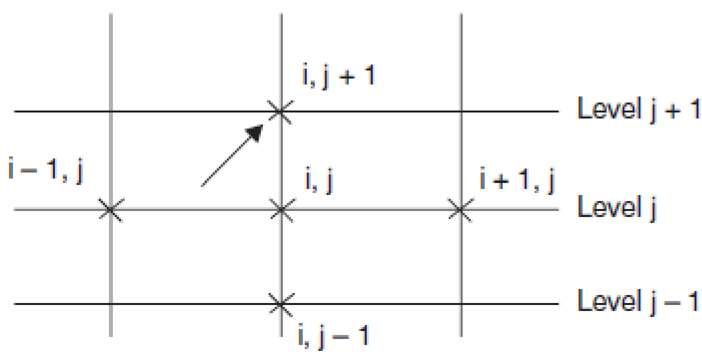

Figure 2: Nods in explicit

Remark: -if α=1 then (4) becomes

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{2.5}$$

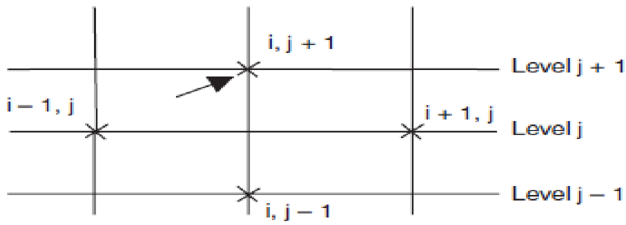

Figure 3: Nodes in explicit method for $\alpha = 1$

Computational procedure

Since the explicit method (24) or (2.5) is of the three levels, we need data on two-time level t=0 and t=k to start the computation. The boundary conditions u(0, t), u(L, t),t>0 gives the solution of all nodal points on the line x=0 and x=L for all time levels. We choose the values of k and h depending on the equation, this gives the value of α. The initial condition u(x,0) = f(x) gives the solution at all the nodal points on the initial line (level 0). The values required on the level t=k is obtained by writing a suitable approximation to the initial condition

$$\frac{\partial u}{\partial x}(x,0) = g(x) \tag{2.6}$$

If we write the central difference approximation, we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{2k}\left(u_{i,1} - u_{i,-1}\right) = \mathrm{g}(x_i) \tag{2.7}$$

Solving for $u_{i,-1}$ from (2.7) we get

$$u_{i,-1} = u_{i,1} - 2k\mathrm{g}(x_i) \tag{2.8}$$

Now we can use the method (2.4) or (2.5) at the nodes on the level t=k, that is, for j=0 we get

$$u_{i,1} = 2(1-\alpha)u_{i,0} + \alpha\left[u_{i+1,0} + u_{i-1,0}\right] - u_{i,-1} \tag{2.9}$$

The external point $u_{i,-1}$that is introduced in the above equation is eliminated by using the above relation (2.8).

$$u_{i,1}=2(1-\alpha)u_{i,0} + \alpha\left[u_{i+1,0}+u_{i-1,0}\right]-[u_{i,1}-2kg(xi)]$$

Or $$2u_{i,1}=2(1-\alpha)u_{i,0} + \alpha\left[u_{i+1,0}+u_{i-1,0}\right]+2kg(xi) \quad (2.10)$$

This gives the values of all nodal points on the level t=k

For example, if the initial condition is prescribed as

$$\frac{\partial u}{\partial x}(x,0)=0$$

We get (2.8), $u_{i,-1}=u_{i,1}$

The formula from (2.9) becomes

$$u_{i,1}=(1-\alpha)u_{i,0} + \frac{\alpha}{2}\left[u_{i+1,0}+u_{i-1,0}\right] \quad (2.11)$$

For= 1, the method implies to

$$u_{i,1}=\frac{1}{2}\left[u_{i+1,0}+u_{i-1,0}\right] \quad (2.12)$$

Thus the solution of all nodal points on level 1 are obtained. For t>k, that is for j≥1. We use the method (2.4) or (2.5).The computation is repeated for the required number of steps. If we perform m steps of computation, then we have to compute the solution up to time t_m= mk.

Example.3: solve the wave equation $u_{tt}=u_{xx}$ with the boundary conditions u(0,t)=0=u(1,t) and the initial conditions $u_t(x,0)=0, u(x,0)=\frac{x}{2}(1-x)$

a. For k=0.1 and h=0.25, 0.2, 0.125 and 0.1for five time level
b. For h=0.1 and k= 0.125, 0.1for five-time level

Solution: The exact solution is given by:

$$u(x)=\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi t\ \sin(2n+1)\pi x}{(2n+1)^3}$$

We can observe that even though it is an infinite series the analytic solution of the given wave equation at t=5 is zero.

A. For k=0.1 and h=0.25, 0.2, 0.125 and 0.1for five-time level

Table 1**:** the computational results for example 3 for h = 0.25 and k=0.1

x / t	0.00	0.25	0.50	0.75	1.00
0.0	0.00000	0.09730	0.12500	0.09730	0.00000
0.1	0.00000	0.08875	0.12000	0.08875	0.00000
0.2	0.00000	0.07455	0.10500	0.07455	0.00000
0.3	0.00000	0.05329	0.08026	0.05329	0.00000
0.4	0.00000	0.02782	0.04688	0.02782	0.00000
0.5	0.00000	0.00095	0.00710	0.00095	0.00000
Exact result at t=0.5	0.00000	0.00000	0.00000	0.00000	0.00000
Absolute error at t=0.5	0.00000	0.00095	0.00710	0.00095	0.00000

Table 2: The computational results for example 3 for h = 0.2 and k=0.1

x / t	0.00	0.20	0.40	0.60	0.80	1.00
0	0.000	0.080	0.120	0.120	0.080	0.000
0.1	0.000	0.075	0.115	0.115	0.075	0.000
0.2	0.000	0.061	0100	0100	0.061	0.000
0.3	0.000	0.042	0.075	0.075	0.042	0.000
0.4	0.000	0.020	0.042	0.042	0.020	0.000
0.5	0.000	0.001	0.004	0.004	0.001	0.000
Analytic result at t=0.5	0.000	0.000	0.000	0.000	0.000	0.000
Absolute error at t=0.5	0.000	0.001	0.004	0.004	0.001	0.000

Table 3: The computational results for example 3 for h = 0.125 and k=0.1

t \ x	0.000	0.125	0.250	0.325	0.500	0.625	0.750	0.875	1.000
0	0.000	0.550	0.094	0.117	0.125	0.117	0.094	0.550	0.000
0.1	0.000	0.500	0.089	0.112	0.120	0.112	0.089	0.500	0.000
0.2	0.000	0.038	0.074	0.097	0.105	0.097	0.074	0.038	0.000
0.3	0.000	0.025	0.051	0.072	0.080	0.072	0.051	0.025	0.000
0.4	0.000	0.012	0.025	0.038	0.045	0.038	0.025	0.012	0.000
0.5	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000
Analytic result at t=0.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Absolute error at t=0.5	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000

Table 4: The computational results for example 3 for h = 0.1 and k=0.1

t \ x	0.0	0.1	0.2	0.3	0.4	0.5
0	0.000	0.045	0.080	0.105	0.120	0.125
0.1	0.000	0.040	0.075	0.100	0.115	0.120
0.2	0.000	0.030	0.060	0.085	0.1000	0.105
0.3	0.000	0.020	0.040	0.060	0.075	0.080
0.4	0.000	0.001	0.020	0.030	0.040	0.045
0.5	0.000	0.000	0.000	0.000	0.000	0.000
Analytic result at t=0.5	0.000	0.000	0.000	0.000	0.000	0.000
Absolute error at t 0.5	0.000	0.000	0.000	0.000	0.000	0.000

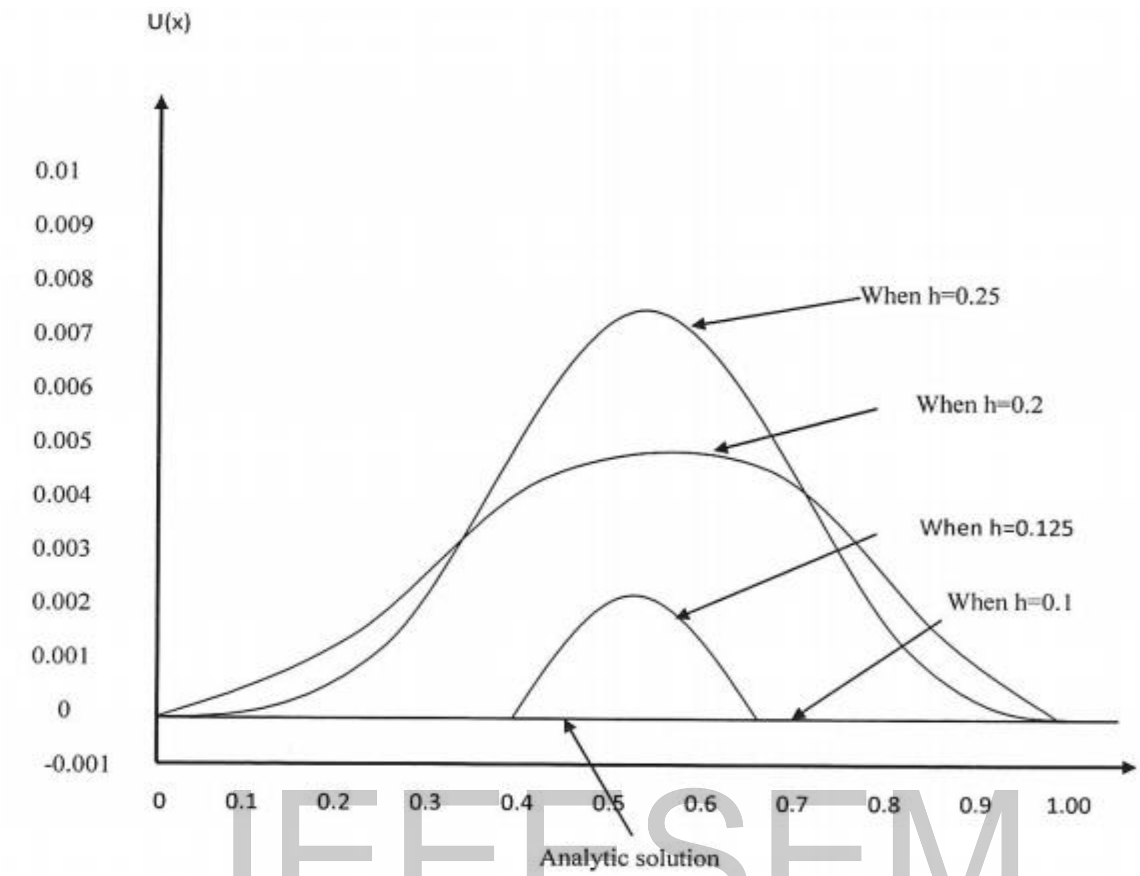

Figure 4: The relation between the size of h and k with the exact solution

From the above results we can observe that when the values of h and k are small enough the approximation solution approaches to the exact solution at t=0.5. From table 4 there is a complete agreement between the approximation solution and the analytic solution at t=0.5 for h=k=0.1.

5.3 Implicit method

We write the following approximation at (x_i, t_j)

Using the central difference method, we can write the approximation as

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} \approx \frac{1}{\mathrm{k}^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] \tag{2.13}$$

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} \approx \frac{1}{2\mathrm{h}^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] \tag{2.14}$$

Form equation (2.1), (2.13) and (2.14) we get

$$\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] = \frac{1}{2h^2}\left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}\right]$$

Or $-\frac{\alpha}{2}\left[u_{i+1,j+1} + u_{i-1,j+1}\right] + (1+\alpha)u_{i,j+1} = \frac{\alpha}{2}\left[u_{i+1,j-1} + u_{i-1,j-1}\right] + 2u_{i,j} - (1+\alpha)u_{i,j-1}$ (2.15)

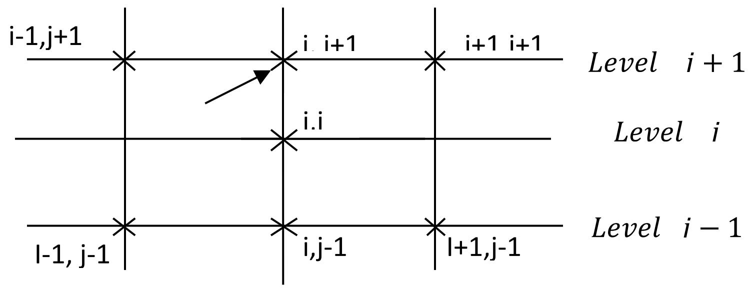

Figure 5: Nodes in implicit method

Computational procedure

The initial condition u(x,0)=f(x) gives the solution at all the nodal points on the initial line (level 0).The boundary conditions u(o, t)=g(x),u(l, t)=h(x),t>0 give the solution at all the nodal points on the line x=0 and x=l for all time levels. We choose the values of h and k. This gives the mesh ratio parameter α.

On level one, we use the same approximation as the case of explicit method, that is, we approximate

$$u_{i,-1} = u_{i,1} - 2kg(x_i) \quad (2.16)$$

Now, we apply the finite difference method (2.15) on level one

For j=0, from (2.15) and (2.16) we get we obtain

$$2u_{i,1} - \alpha(u_{i+1,1} - u_{i,1} + u_{i-1,1}) = 2u_{i,0} + 2kg_i - k\,\alpha\,(g_{i+1} + 2g_i + g_{i-1}) \quad (2.17)$$

If the initial condition is u_t(x, 0) =0, the method simplifies as

$$2u_{i,1} - \alpha(u_{i+1,1} - u_{i,1} + u_{i-1,1}) = 2u_{i,0} \quad (2.18)$$

The right hand side in (2.17) or (2.18) is computed. For i=1,2,3,. . . .,M-1,we obtain system of equations for $u_{1,1}$, $u_{2,1}$, . . ., $u_{M-1,1}$.The system of equations is solved to obtain the values of all nodal points on the time level one. For j>0, we obtain the method 2.15 and solve the system of equations on each mish line. The computations are repeated for the required number of steps. If we perform m steps of computation, and then we have to compute the solution up to t_m=mk.

To make it more clear let us see the following example

Example: 4 solve the wave equation u_{tt}=u_{xx}, 0≤x≤1, Subject to the condition

u(x,0)=sin(πx),u_t(x,0)=0, u(0,t)=u(1,t)=0,t>0

Use the implicit method with h=k=$\frac{1}{4}$

Solution: we have c=1, h=k=$\frac{1}{4}$, then $\alpha = \frac{c^2k^2}{h^2}$=$\frac{(0.25^2)}{0.25^2}$=1

For α=1, then (2.15) can becomes

$$-\frac{1}{2}\left[u_{i+1,j+1}+u_{i-1,j+1}\right]+2(u_{i,j+1})=\frac{1}{2}\left[u_{i+1,j-1}+u_{i-1,j-1}\right]+2u_{i,j}-2(u_{i,j-1}) \qquad j=0,1;\ i=1,2,3 \qquad (2.19)$$

The boundary conditions give the values $u_{0,j}=0=u_{4,j,}$for all j

The initial condition u(x, 0) =sin (πx), gives the values

$u_{0,0}$=0 , $u_{1,0}$ = sin (π/4)= (1/$\sqrt{2}$)

$u_{2,0}$= sin (π/2)= 1

$u_{3,0}$= sin (3π/4)= (1$\sqrt{2}$), $u_{4,0}$= 0

The initial condition u_t(x, 0) =0, gives the values

$$u_{i,-1,}=u_{i,1}$$

Therefore, for j=0 we get the equation

$0.5u_{i,-1} + 2u_{i,1} - 0.52u_{i+1,1} = \frac{1}{2}\left[u_{i+1,-1} + u_{i-1,-1}\right] + 2u_{i,0} - 2u_{i,-1}$

Or $-u_{i-1,1} + 4u_{i,1} - u_{i+1,1} = 2u_{i,0}$

We have the following equations for j=0 and j=1 we have the following results

Under the give table below are the computation results from FORTRAN program for different levels.

Table 5. The computational result of example 4 using implicit method for h=k=0.25

j \ i	0	1	2	3	4
0	0.0000000	0.7071068	1.0000000	0.7071068	0.0000000
1	0.0000000	0.5469182	0.7734591	0.5469182	0.0000000
2	0.0000000	0.1389309	0.1964779	0.1389309	0.0000000
3	0.0000000	-0.3320035	-0.4695238	-0.3320035	0.0000000

5.4 Some application examples

Simple and historically important example of a problem that includes the wave equation is provided by the study of the vibrating of a string, like a violin or guitar string. We set up the coordinate system as shown in the figure below. Consider an elastic string stretched between two pegs, as on a guitar we have to describe the motion of the string if it is given a small displacement and released to vibrate in a plane. Place the string along the x-axis from 0 to L and assume it vibrates in the x, y plane. We want a function u(x, y) at any time t>0, the graph of the function u=(x, t) of x, is the shape of the string at that time. Thus u(x, t) allows us to take a snapshot of the string at any time, showing it as a curve in the plane. For this reason u(x, t) is called the position function for the string . Let us neglect damping force such that air resistance and the weight of the string and assume that the tension T(x, t) in the string always acts tangentially to the string, and individual particles of the string move only vertically. Also assume that mass per unit length is constant.

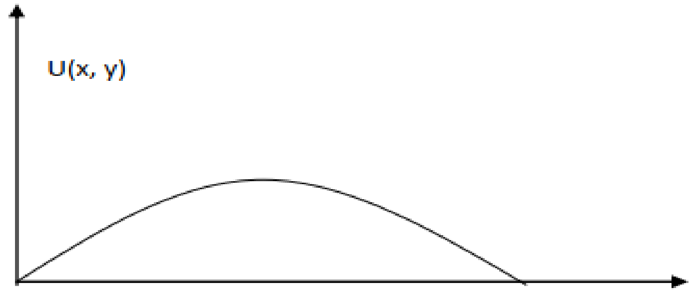

Fig : 6. String fixed at two ends

Example: 5 The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies an equation u_{tt}=4u_{xx}, with the boundary conditions u=0 at x=0, t>0 and u=0 at x=4,t>0 and initial conditions u=x(4-x)and u_t=0 at t=0,0≤x≤4.solve this equation numerically for one half period of vibration, taking h=1 and k= $\frac{1}{2}$.

Solution: Computational result of example 5, for h=1 is given in the table below

Table 6: Computational result of example 5, for h=1

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	-3	-2	0
4	0	-3	-2	-3	0

Example: 6 A string 80 cm long weighting 1N/m is stretched with a tension of 400N, at a point 20 cm from one end and it is pulled o.6cm from equilibrium and released. Find the displacement along the string as function of time g(x) =0(no velocity)

Solution: T= 400N; g=9.81m/sec^2 ρ =1N/M

$$c^2 = T/\rho = \frac{400x9.81}{1} = 3924$$

Since Δx=0.1 we can find Δt such that α=1

$$\alpha=\frac{c^2\Delta t}{\Delta x^2}=\frac{3924x\Delta t}{0.01}=1$$

$$\Delta t= 0.001596$$

Since we are choosing the value of α to be one, the difference equation for the given equation is given by

$$u_{i,j+1}= u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

Depending on the given information from the problem, at the time t=0, the string has the following shape

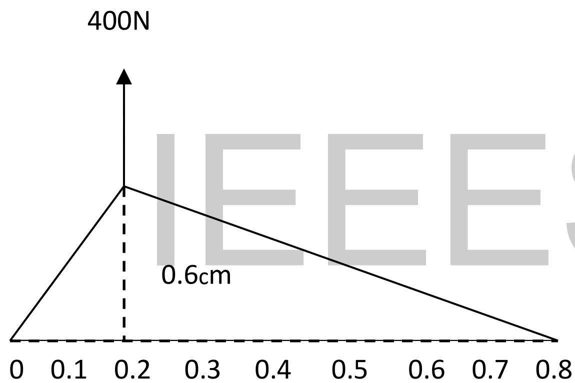

Figure 7: A stretched strings

By using the Pythagoras relation from the above figure we can calculate the values

$u_{i,o}$,i=1, 2, 3, 4, 5, 6, 7

$u_{0,0}$= 0, $u_{1,0}$= 0.3, $u_{2,0}$= 0.6, $u_{3,0}$= 0.5, $u_{4,0}$= 0.4,

$u_{5,0}$= 0.3, $u_{6,0}$= 0.2, $u_{7,0}$= 0, $u_{8,0}$= 0

These are the entries for the first row

Since u_t(x, 0) =g(x) = 0, we have

$\frac{u_{i,j+1}-u_{i,j-1}}{2k}$=0, when j=o gives $u_{i,1}= u_{i,-1}$

Thus the entries of the second row are the same as the entries of the first row

Putting j=0 in $u_{i,j+1}=u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$, we have

$$u_{i,1}=u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

$$=u_{i-1,0} + u_{i+1,0} - u_{i,1}$$

This implies $u_{i,1}=\frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking i=1, 2, 3, 4, 5, 6, 7 successively we obtain

$u_{1,1}=\frac{1}{2}(u_{0,0} + u_{2,0})$= 0.5(0+0.6) =0.3

$u_{2,1}=\frac{1}{2}(u_{1,0} + u_{3,0})$= 0.5(0.3+0.5) =0.4

$u_{3,1}=\frac{1}{2}(u_{2,0} + u_{4,0})$=0.5(0.6+0.4) =0.5

$u_{4,1}=\frac{1}{2}(u_{3,0} + u_{5,0})$=0.5(0.5+0.3) =0.4

$u_{5,1}=\frac{1}{2}(u_{4,0} + u_{6,0})$=0.5(0.4+0.2) =0.3

$u_{6,1}=\frac{1}{2}\left(u_{5,0} + u_{7,0}\right)$=0.5(0.3+0.1) =0.2

$u_{7,1}=\frac{1}{2}\left(u_{6,0} + u_{8,0}\right)$=0.5(0.2+0) =0.1

These are the entries of the second row

Similarly, the position of the string for each fraction of time is given in the following table below.

Table 7: The computation result for each step for example 6

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.000000	0	0.3	0.6	0.5	0.4	0.3	0.2	0.1	0
0.001596	0	0.3	0.4	0.5	0.4	0.3	0.2	0.1	0
0.003192	0	0.1	0.2	0.3	0.4	0.3	0.2	0.1	0
0.004788	0	-0.1	0	0.1	0.2	0.3	0.2	0.1	0
0.006384	0	-0.1	-0.2	-0.1	0	0.1	0.2	0.1	0
0.007980	0	-0.1	-0.2	-0.3	-0.4	-0.3	-0.2	-0.1	0
0.009576	0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.4	-0.3	0
0.011172	0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.3	0
0.012768	0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.5	0
0.014364	0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	0

6. Conclusion

In this report, we discussed the finite difference approximation for hyperbolic partial differential equations. In this method both the explicit and implicit methods of finite difference approximations were discussed. As we have seen from the computation results, finite difference method of solving differential equations is mesh size dependent. That is the method of the accuracy increases when the mesh size is small enough. The computation result also indicates that, using implicit finite difference method to solve hyperbolic partial differential equations gives a better approximation than explicit finite difference approaches.

Reference

[1].Conte, S. D., & De Boor, C. (1980). *Elementary numerical analysis: An algorithmic approach* (3d ed). McGraw-Hill.

[2]. Hoffman, J. D. (2001a). *Numerical methods for engineers and scientists* (2. ed., rev.expanded). CRC Press.

[3]. Hoffman, J. D. (2001b). *Numerical methods for engineers and scientists* (2nd ed., rev.expanded). Marcel Dekker.

[4].O'Neil, P. V. (2007). *Advanced engineering mathematics: International student edition*. Thomson.

[5]. Press, W. H. (Ed.). (1996). *FORTRAN numerical recipes* (2nd ed). Cambridge University Press.