IEEE-SEM, Volume 12, Issue 1, January 2024
ISSN 2320-9151

EFFICIENT SOLUTION OF NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS: THE METHOD OF UNDETERMINED COEFFICIENT APPROACH

Udoaka Otobong G.
otobongawasi@aksu.edu.ng
Department of Mathematics
Akwa Ibom State University, Nigeria

Etim Uduak James
uduakjamesetim@gmail.ocom
Department of Mathematics
Akwa Ibom State University, Nigeria

Udoakpan I. u.
itoroubom@yahoo.com
Department of mathematics and Statistics, University of Port Hacourt

Abstract

This work deals with the application of the "Method of Undetermined Coefficient" in solving nonhomogeneous linear differential equations with constant coefficients as an alternative to the method to "Variation of Parameters". Its treatment covers up to the fourth-order nonhomogeneous linear DEs. It has carefully explained the steps and procedures involved in finding the complementary and particular solutions which are usually combined to obtain the general solutions. It also discusses the limitations or the extent to which this method under consideration can be applied.

KEYWORDS: Method of Undetermined Coefficients, Nonhomogeneous Linear Differential Equations, Constant Coefficients, Variation of Parameters, General Solutions, Fourth-order Differential Equations

1. INTRODUCTION

Solving nonhomogeneous linear differential equations (DEs) with constant coefficients is a fundamental task in the realm of differential equations. [1] Foundational work provides insights into the theoretical aspects of ordinary differential equations, including discussions on undetermined coefficients and their application in solving nonhomogeneous linear differential equations. Ince's classic text [2] is a valuable resource that discusses various methods for solving ordinary differential equations, offering historical context and foundational knowledge on undetermined coefficients. [3] Widely used textbook provides practical insights into solving differential equations, including the undetermined coefficients method, making it accessible for students and practitioners. [4]'s introductory book serves as a practical guide, covering the undetermined coefficients method and its application in solving nonhomogeneous linear differential equations. While focused on applications in mathematical biology, Murray's [5] work demonstrates the undetermined coefficients method in specialized contexts, showcasing its versatility beyond traditional differential equations. [6] Article introduces the MATLAB ODE Suite, a practical tool for solving ordinary differential equations, including the undetermined coefficients method. [7] Book provides a comprehensive overview of nonlinear ordinary differential equations, discussing various solution methods, including undetermined coefficients. This work presents an in-depth exploration of the "Method of Undetermined Coefficient" as an alternative approach to solving such equations, offering efficiency and clarity in comparison to the conventional "Variation of Parameters" method. The focus extends to fourth-order nonhomogeneous linear DEs, providing a comprehensive understanding of the application of this method. See [9] to read about weak solutions of nonlinear boundary value problems of partial differential equations using variational method. Also see [10] and [11] for further studies.

2. METHOD

The Method of Undetermined Coefficients is used to find a particular solutiony_P of a nonhomogeneous linear differential equation (DE):

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + a_{n-2}(x)y^{(n-2)}(x) + \cdots + a_1(x)y'(x) + a_o(x)y(x) = g(x); \qquad x \in I, \tag{1}$$

where the unknown function is $g(x)$ and the coefficients of the functions are $a_k(x), 0 \leq k \leq n$, after the complementary solution y_c of the associated homogeneous DE (1),

has been obtained,

the general solution of an n^th^ – order linear DE has the form:

$$y(x) = y_h(x) + y_p(x), \qquad x \in I \tag{2}$$

where $y_h(x)$ is an n-parameter family of solutions of the linear and homogeneous DE of (1) and $y_p(x)$ is a particular solution of the non-homogeneous DE (1). The n-parameter family of solutions, $y_h(x)$ has to be determined as

$$y_h(x) = c_1 y_1(x) + c_2 y_2 \ldots + c_n y_n(x) \tag{3}$$

Where $y_1(x), y_2(x), \ldots, y_n(x)$ are the solutions of the linear and homogeneous DE (1).

The cardinal fact in the Method of Undetermined Coefficients is that, it is a *conjecture*, that is, *an educated guess*, about the form of y_p motivated by the kinds of functions that constitute the input function or the nonhomogeneous part of the equation, $g(x)$. Hence, the method is sometimes called *the Method of Educated Guess*.

The Complementary Solution 2.1

The methods and techniques for obtaining the complementary solution y_c are explained below.

Given the homogeneous linear DE (1), we substitute y = $e^{\lambda x}$ to get the characteristic equation as follows:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0 \tag{4}$$

To facilitate a quick understanding of (4), let's consider a second-order nonhomogeneous linear DE with constants as coefficients:

$ay'' + by' + cy = g(x),$ (5)

where, a, b *and* c are constants.

The corresponding homogeneous equation is:

$ay'' + by' + cy = 0.$ (6)

Substituting y = $e^{\lambda x}$ and its first two derivatives into (6) yields:

$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$

Factoring out $e^{\lambda x}$, we obtain

$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0.$

Since $e^{\lambda x} \neq 0$, it follows that

$a\lambda^2 + b\lambda + c = 0.$ (7)

Equation (7) above is the characteristic equation of (6).

Solving (7) quadratically, we obtain two characteristic roots:

$\lambda_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ *and* $\lambda_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

There are three forms of the general solution of (6) corresponding to the three cases of the characteristic roots:

Case I: λ_1 and λ_2 are real and distinct if $(b^2 - 4ac > 0)$

Case II: λ_1 and λ_2 are real and equal if $(b^2 - 4ac = 0)$

Case III: λ_1 and λ_2 are complex conjugates if $(b^2 - 4ac < 0)$

(Case I): Distinct Real Roots$(b^2 - 4ac > 0)$ 2.2

By the assumption that the characteristic equation (7) has two real roots λ_1 and λ_2, we obtain two linearly independent solutions y = $e^{\lambda_1 x}$ and y = $e^{\lambda_2 x}$, respectively, on the interval $(-\infty, \infty)$. These two independent solutions form a fundamental set. Therefore, the general solution of (6) on this interval is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (8)$$

(Case II): Multiple Real Roots($b^2 - 4ac = 0$) 2.3.

When $\lambda_1 = \lambda_2$, we necessarily find only one exponential solution $y_1 = e^{\lambda_1 x}$. From the quadratic formula:

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

we find that $\lambda_1 = -\frac{b}{2a}$ (because $b^2 - 4ac = 0$).

It follows that we can construct a second solution y_2 by the Reduction of Order Algorithm:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{\left(y_1(x)\right)^2} dx.$$

We begin by writing (6) in standard form, thus:

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0. \ (9)$$

Comparing (9) with $y'' + P(x)y' + Q(x)y = 0$, we notice that P(x) is $= \frac{b}{a}$.

Then the second solution is

$$y_2 = e^{\lambda_1 x} \int \frac{e^{2\lambda_1 x}}{e^{2\lambda_1 x}} dx.$$

$$= e^{\lambda_1 x} \int dx.$$

$$= x e^{\lambda_1 x}$$

It can be shown that $y_1 = e^{\lambda_1 x}$ and $y_2 = xe^{\lambda_1 x}$ are linearly independent solutions, and so, they form a fundamental set. Therefore, the general solution of (6) is:

$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$. (10)

(Case III): Conjugate Complex Roots ($b^2 - 4ac < 0$) 2.4

When $b^2 - 4ac < 0$, the characteristic equation (6) has two complex roots, which are conjugates. These roots are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where $\alpha, \beta > 0$, and $i^2 = -1$.

As in case I, these roots give two linearly independent solutions $y_1 = e^{(\alpha + i\beta)X}$ and

$y_2 = e^{(\alpha - i\beta)X}$.

Consequently, the linear combination:

$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$ (11)

will be a general solution.

However, in practice, we prefer to work with real functions rather than complex exponentials. To achieve this, we make use of Euler's Formula:

` $e^{i\theta} = \cos\theta + i\sin\theta,$ (12)

$where\ \theta\ is\ any\ real\ number.$

Expanding (11), we obtain:

$y = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$

Factoring out $e^{\alpha x}$, we get:

$y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$. (13)

From Euler's Formula,

$e^{i\beta x} = cos\beta x + isin\beta xande^{-i\beta x} = cos\beta x - isin\beta x$ (14)

Plugging (14) in (13) yields:

y = $e^{\alpha x}\{c_1(cos\beta x + isin\beta x) + c_2(cos\beta x - isin\beta x)\}$

= $e^{\alpha x}\{c_1 cos\beta x + ic_1 sin\beta x + c_2 cos\beta x - ic_2 sin\beta x\}$

= $e^{\alpha x}\{(c_1 + c_2)cos\beta x + i(c_1 - c_2)sin\beta x\}$.

Now, if we let $A = c_1 + c_2$and $B = i(c_1 - c_2)$, we get:

y = $e^{\alpha x}(Acos\beta x + Bsin\beta x)$. (15)

Equation (15) above is the general solution to (6) when the characteristic roots are complex conjugates.

The Particular Solution 2.5

Having examined the methods for determining the complementary solution$y_c$we now proceed to the particular solution $y_P$and the methods for obtaining it.

As earlier mentioned, there are majorly two methods for finding the particular solution y_p.These are the Method of Variation of Parameters and the Method of Undetermined Coefficients. In this paper, we are considering the Method of Undetermined Coefficients.

The Method of Undetermined Coefficients is one particularly simple-minded, yet very effective method for finding particular solutions to nonhomogeneous linear DEs with constant coefficients. The method is also known as the Method of *"Educated Guess."*Hence, it is based on making *"good guesses"* regarding these particular solutions. And, as always, *"good guessing"* is usually aided by a thorough understanding of the problem. The method usually works best if the problem is simple enough. Fortunately, a great many nonhomogeneous DEs of interest are sufficiently simple. Here, simplicity is defined in terms of the class of DEs one is working with. For a DE to be fit to be solved by the Method of Undetermined Coefficients, it must be a nonhomogeneous linear DE with constant coefficients, whose input function is a constant, a polynomial function, an exponential function, a sine function, a cosine function or a linear combination of such functions.

As with the complementary solution, we shall remain within the ambit of this work by considering up to Fourth-order DEs.

Suppose we wish to find a particular solution to a nonhomogeneous Second-order DE (5). If $g(x)$ is among the aforementioned functions, and the coefficients $a, b$and c, are constants, then we might be able to make a good guess as to what sort of function $y_p(x)$ yields $g(x)$. The guess will involve specific functions and some constants that can be determined by plugging the assumed formula for $y_p(x)$ into the DE, and solving the resulting algebraic equation(s) for these constants (provided the initial guess was good).

In Table 2.6 and 2.7, we illustrate some specific examples of $g(x)$ alongside the corresponding forms of the particular solution. We are, of course, assuming that no function in the particular solution $y_p$is duplicated by a function in the complementary function y_c.

In general, the following table shows the forms of particular integrals of different functions.

Table 2.6

Type	**Straightforward Cases** ***Try as particular integral***	**Snag Cases** ***Try as particular integral***
(a) f(x) = a constant	$y_p = k$	$y_p = kx$ (used when C>F contains a constant)
(b) f(x) = polynomial (i.e. $f(x) = L + Mx + Nx^2 + \ldots$ $(where\ any\ of\ the\ coefficier$	$y_p = a + bx + cx^2 + \cdots$	
(c) f(x) = an exponential function. (i.e. *f(x)* $Ae^{\alpha x}$)	$Y_p = ke^{\alpha x}$	*(i)* $y_p = kxe^{\alpha x}$ (used when $e^{\alpha x}$appears in the C.F). *(ii)* $y_p = kx^2e^{\alpha x}$ (used when e^{ax} and

		xe^{ax} both appears in the C.F)
(d) f(x) = a sine or cosine function (i.e. f(x) = a sin*p*x + b cos*p*x, where a and b may be zero)	y_p =A sin*p*x + B cos*p*x	y_p = x(A sin px + B cos*p*x) (used when sin*p*x and/or cos*p*x appears in the C.F.)
(e) *f(x)* = a sum e.g. *(i)* $f(x) = 4x^2 - 3\sin 2x$ (ii) $f(x) = 2 - x + e^{3x}$	(i) $y_p = (Ax^2 + Bx + C) + (D\sin 2x + E\cos 2x)$ (ii) $y_p = Ax + B + Ce^{3x}$	
(f) *f(x)* a product, e.g. $f(x) = 2e^x \cos 2x$	$y_p = e^x(A\sin 2x + B\cos 2x)$	

Table 2.7: Illustrative examples are shown in the table below:

Trial Particular Solutions

g(x)	**Form of y_p**
1. 4 (any constant)	A
2. 3x + 8	Ax + B
3. $9x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 2$	$Ax^3 + Bx^2 + Cx + D$
5. sin 3x	A cos3x + B sin 3x
6. cos6x	A cos6x + B sin 6x
7. e^{4x}	Ae^{4x}
8. $(6x - 4)e^{8x}$	$(Ax + B)e^{8x}$
9. x^3e^{3x}	$(Ax^3 + Bx^2 + Cx + D)e^{3x}$
10. e^{6x} cos 6x	Ae^{6x} cos 6x + Be^{6x} sin 6x
11. $10x^2$ sin 2x	$(Ax^2 + Bx + C)\cos 2x + (Dx^2 + Ex + F)\sin 2x$
12. xe^{5x} cos 4x	$(Ax + B)e^{5x}\cos 4x + (Cx + D)e^{5x}\sin 4x$

3. EXAMPLES

Using the techniques considered above and the methods, we solve the following problems:

Example 3.1: Write the form of the particular solution $y_p(x)$ of

$$y'' + p(x)y' + \; q(x)y = 2\,\mathrm{Sin}(3x) + 4\,Cos(3x) \quad (16)$$

Solution:

We consider the R.H.S of the equation: $2\,\mathrm{Sin}(3x) + 4\,Cos(3x)$,

The particular solution is guessed as:

$$y_p(x) = A\,Sin(3x) + B\,Cos(3x) + C\,Cos(3x) + D\,Sin(3x) \qquad (17)$$

$$y_p(x) = (A + D)Sin(3x) + (B + C)Cos(3x)$$

$$y_p(x) = E\,Sin(3x) + F\,Cos(3x), \quad where (A + D) = E \; and \; (B + C) = F \qquad (18)$$

Example 3.2: Write the form of the particular solution y_p of

$$y'' + p(x)y' + \; q(x)y = 2x^2 - 3Sin(5x) + 1. \qquad (19)$$

Solution:

The R.H.S of the given equation is $2x^2 - 3Sin(5x) + 1$.

Now, we guess the particular solution as

$$y_p = \; Ax^2 + Bx + C + DSin(5x) + E\,Cos(5x) \qquad (20)$$

Example 3.3: Write the form of the particular solution of

$$y'' + p(x)y' + \; q(x)y = 4\,e^{2x} - 3x\,e^{-x} + 2e^{-x} \qquad (21)$$

Solution: Assume no term of y_p are duplicated in y_c

Then, $y_p(x) = Ae^{2x} + (Bx + C)e^{-x} + De^{-x}$

$$y_p(x) = \; Ae^{2x} + Bxe^{-x} + Ce^{-x} + De^{-x}$$

$$y_p(x) = \; Ae^{2x} + Bxe^{-x} + (C + D)e^{-x}$$

$$y_p(x) = \; Ae^{2x} + Bxe^{-x} + Ke^{-x} \qquad (where\; (C + D) = K) \qquad (22)$$

Example 3.4: Determine the form of the particular solution of the DE:

$y'' + 9y = 4t^3 Sin(3t)$ (23)

Solution:

$\boldsymbol{y_p} = (\mathrm{A}t^3 + Bt^2 + Ct + D)Sin(3t) + (Et^3 + Ft^2 \; Gt + H)Cos(3t)$(23)

For the homogeneous equation,

$y'' + 9y = 0$ (24)

$\Rightarrow r^2 + 9 = 0 \;\Rightarrow\; r^2 = -9 \qquad \Rightarrow\; r = 3i \; or - 3i$

$The\ complementary\ solution, y_h = C_1 Cos(3t) + C_2\, Sin(3t)$ (25)

Hence, the particular solution is

$y_p = t(At^3 + Bt^2\; Ct + D)Sin\,(3t) + t(Et^3 + Ft^2 + Gt + H)Cos(3t)$

$y_p = (At^4 + Bt^3\; Ct^2 + Dt)Sin\,(3t) + t(Et^4 + Ft^3 + Gt^2 + Ht)Cos(3t)$ (26)

Example 3.5: Find the general solution of $\boldsymbol{y'' + 4y' + 3y = 3x}$.

Solution:

$y'' + 4y' + 3y = 3x$ (27)

As we already know, the first thing to do is to obtain the complementary function. We begin by getting the corresponding homogenous equation of (27):

$y'' + 4y' + 3y = 0.$ (28)

Substituting y = $e^{\lambda x}$and its first and second derivatives into (28) gives

$\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 3e^{\lambda x} = 0$

Factoring out $e^{\lambda x}$, gives

$e^{\lambda x}(\lambda^2 + 4\lambda + 3) = 0$

Since $e^{\lambda x} \neq 0$, it follows that the characteristic equation of (28) is:

$$\lambda^2 + 4\lambda + 3 = 0. \quad (29)$$

By the method of factorization, we obtain:

$$\lambda^2 + 3\lambda + \lambda + 3 = 0$$

$$\lambda(\lambda + 3) + 1(\lambda + 3) = 0$$

$$(\lambda + 1)(\lambda + 3) = 0$$

$$\therefore \quad \lambda = -1, -3$$

Therefore, the complementary solution is:

$$y_c = c_1 e^{-x} + c_2 e^{-3x}. \quad (30)$$

Next, we obtain the particular solution. From (27), since $g(x)$ is a linear polynomial, it suffices to make a conjecture of y_p as:

$$y_p = Ax + B \quad (31)$$

Taking the derivative of (49) twice yields

$$y_p' = A \text{ and } y_p'' = 0 \quad (32)$$

Substituting (31) and (32) into (27), we have

$$4A + 3(Ax + B) = 3x$$

$$4A + 3Ax + 3B = 3x$$

Comparing the coefficients of like terms, we get

x: $3A = 3$ (i)

CT: $4A + 3B = 0$ (ii)

From (i),

A = 1 (iii)

Putting 1 for A in equation (ii) yields

4(1) + 3B = 0

∴ B = – $\frac{4}{3}$ (iv)

Substituting (iii) and (iv) into (31), we obtain a particular solution

$$y_p = x - \frac{4}{3} . \tag{33}$$

The sum of the complementary solution and the particular solution gives us the general solution

$$y = c1e^{-x} + c2e^{-3x} + x - \frac{4}{3}. \tag{34}$$

4. CONCLUSION

The Method of Undetermined Coefficient emerges as a powerful and efficient tool for solving nonhomogeneous linear differential equations with constant coefficients up to the fourth order. This approach simplifies the solution process, providing clear steps for determining both the complementary and particular solutions. While acknowledging its strengths, this work also discusses the limitations and contexts within which the method is most applicable. Overall, this method stands as a valuable alternative in the toolkit of differential equation solvers, offering clarity and effectiveness in its application.

References

[1] Coddington, E. A., & Levinson, N. (1955). Theory of Ordinary Differential Equations. Tata McGraw-Hill Education.

[2] Ince, E. L. (1927). Ordinary Differential Equations. Dover Publications.

[3] Boyce, W. E., & DiPrima, R. C. (2012). Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons.

[4] Tenenbaum, M., & Pollard, H. (1985). Ordinary Differential Equations: An Introduction to the Fundamentals and Applications. Dover Publications.

[5] Murray, J. D. (1993). Mathematical Biology I: An Introduction. Springer.

[6] Shampine, L. F., & Reichelt, M. W. (1997). The MATLAB ODE Suite. SIAM Journal on Scientific Computing, 18(1), 1-22.

[7] Jordan, D. W., & Smith, P. (2007). Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers. Oxford University Press.

[8] Kreyszig, E. (2010). Advanced Engineering Mathematics. John Wiley & Sons.

[9] Etim, U. J, Udoaka O. G, & John, M. N. (2024). Weak Solutions Of Nonlinear Boundary Value Problems Of Partial Differential Equations Using Variational Method**.** *IEEE-SEM, Volume 12, Issue 1. Retrieved from ; https://www.ieeesem.com/researchpaper*

[10] Udoaka O. G. and Frank E. A.,(2022). Finite Semi-group Modulo and Its Application to Symmetric Cryptography. INTERNATIONAL JOURNAL OF PURE MATHEMATICS DOI: 10.46300/91019.2022.9.13.

[11] Udoaka, O. G.,(2022). Generators and inner automorphism. THE COLLOQUIUM -A Multi disciplinary Thematc Policy Journal www.ccsonlinejournals.com Volume 10 , Number 1 , 2022 Pages 102 -111 CC-BY-NC-SA 4.0 International Print ISSN : 2971-6624 eISSN: 2971-6632.